

QUANTIZATION NOISE AND SIGNAL TO QUANTIZATION NOISE RATIO

In PCM

Signal to Quantization Noise Ratio

- General equation for Signal to Noise ratio $= \frac{\text{Signal power}}{\text{Noise Power}}$
- Signal to Quantization Noise ratio $= \frac{S}{N} = \frac{\text{Signal power}}{\text{Error Noise power}}$
- To find above ratio, It is required to find a mathematical expression for both signal power and quantization – error power
- If the Signal power , S denoted by P,
- then it required to find a mathematical expression for the quantization Noise power, N

Derivation equation for quantization Noise power

- Because of quantization, inherent errors are introduced in the quantized signal, this error is called *quantization error*.

- $\epsilon = X_q(nT_s) - X(nT_s)$

- The $X(nT_s)$ is continuous in amplitude in the range: $-X_{\max}$ to $X_{\max}$

- Total amplitude range become $X_{\max} - (-X_{\max}) = 2 X_{\max}$

- If this amplitude range is divided into q levels of quantizer, then the step size δ is

given by:
$$\delta = (X_{\max} - (-X_{\max})) / q = 2 X_{\max} / q$$

- If the signal $x(t)$ is normalized to minimum and maximum values equal to 1, then

$$X_{\max} = 1, -X_{\max} = -1$$

Therefore step size will be,
$$\delta = 2/q$$

continue

- If step size δ is sufficiently small, then it is reasonable to assume that the quantization error ε will be **uniformly distributed random variable**.

- The maximum will be **$\varepsilon_{\max} = \left[\frac{\delta}{2} \right]$**

i.e.
$$-\frac{\delta}{2} \geq \varepsilon_{\max} \geq \frac{\delta}{2}$$

- Thus over the interval $[-\frac{\delta}{2}, \frac{\delta}{2}]$ quantization error is uniformly distributed random variable

$$\varepsilon_{\max} = \left| \frac{\delta}{2} \right| \quad \dots (6.1.13)$$

i.e.
$$-\frac{\delta}{2} \geq \varepsilon_{\max} \geq \frac{\delta}{2} \quad \dots (6.1.14)$$

Thus over the interval $\left(-\frac{\delta}{2}, \frac{\delta}{2}\right)$ quantization error is uniformly distributed random variable.

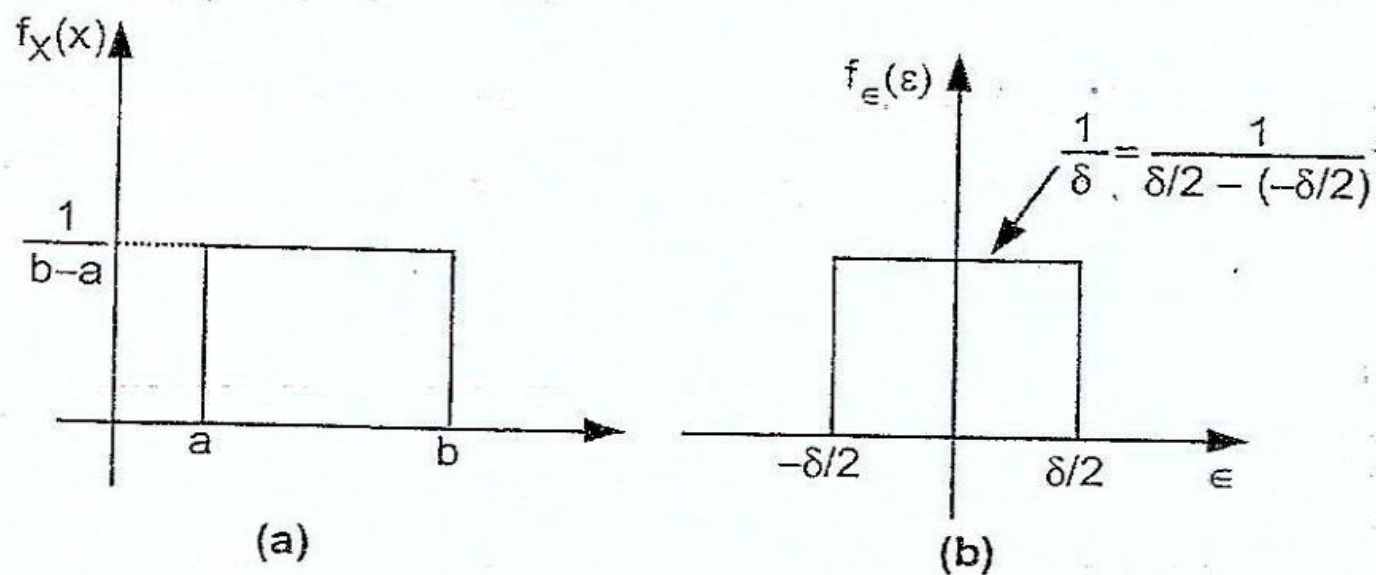


Fig. 6.1.5 (a) Uniform distribution
(b) Uniform distribution for quantization error

In Fig. 6.1.5 (a), a random variable is said to be uniformly distributed over an interval (a, b). Then PDF of 'X' is given by, (from equation of Uniform PDF)

$$\begin{aligned} f_X(x) &= 0 & \text{for } x \leq a \\ &= \frac{1}{b-a} & \text{for } a < x \leq b \\ &= 0 & \text{for } x > b \end{aligned} \quad \dots (6.1.15)$$

Thus with the help of equation 6.1.15 we can define the probability density function for quantization error 'ε' as,

$$\begin{aligned} f_\varepsilon(\varepsilon) &= 0 & \text{for } \varepsilon \leq -\frac{\delta}{2} \\ &= \frac{1}{\delta} & \text{for } -\frac{\delta}{2} < \varepsilon \leq \frac{\delta}{2} \\ &= 0 & \text{for } \varepsilon > \frac{\delta}{2} \end{aligned} \quad \dots (6.1.16)$$

From Fig. 6.1.2 (b) we can see that quantization error 'ε' has zero average value. That is mean ' m_ε ' of the quantization error is zero.

The signal to quantization noise ratio of the quantizer is defined as,

$$\frac{S}{N} = \frac{\text{Signal power (normalized)}}{\text{Noise power (normalized)}} \quad \dots (6.1.17)$$

If type of signal at input i.e., $x(t)$ is known, then it is possible to calculate signal power.

The noise power is given as,

$$\text{Noise power} = \frac{V_{noise}^2}{R} \quad \dots (6.1.18)$$

Here V_{noise}^2 is the mean square value of noise voltage. Since noise is defined by random variable ' ϵ ' and PDF $f_{\epsilon}(\epsilon)$, its mean square value is given as,

$$\text{mean square value} = E[\epsilon^2] = \bar{\epsilon}^2 \quad \dots (6.1.19)$$

The mean square value of a random variable 'X' is given as,

$$\bar{X}^2 = E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x) dx \quad \text{By definition} \quad \dots (6.1.20)$$

Here

$$E[\epsilon^2] = \int_{-\infty}^{\infty} \epsilon^2 f_{\epsilon}(\epsilon) d\epsilon \quad \dots (6.1.21)$$

From equation 6.1.16 we can write above equation as,

$$\begin{aligned} E[\varepsilon^2] &= \int_{-\delta/2}^{\delta/2} \varepsilon^2 \times \frac{1}{\delta} d\varepsilon \\ &= \frac{1}{\delta} \left[\frac{\varepsilon^3}{3} \right]_{-\delta/2}^{\delta/2} \\ &= \frac{1}{\delta} \left[\frac{(\delta/2)^3}{3} + \frac{(\delta/2)^3}{3} \right] \\ &= \frac{1}{3\delta} \left[\frac{\delta^3}{8} + \frac{\delta^3}{8} \right] = \frac{\delta^2}{12} \end{aligned}$$

... (6.1.22)

∴ From equation 6.1.19, the mean square value of noise voltage is,

$$V_{noise}^2 = \text{mean square value} = \frac{\delta^2}{12}$$

When load resistance, $R = 1$ ohm, then the noise power is normalized i.e.,

$$\begin{aligned} \text{Noise power (normalized)} &= \frac{V_{noise}^2}{1} \quad [\text{with } R = 1 \text{ in equation 6.1.18}] \\ &= \frac{\delta^2 / 12}{1} \\ &= \frac{\delta^2}{12} \end{aligned}$$

SE

Thus we have,

$$\begin{aligned} &\text{Normalized noise power} \\ &\text{or Quantization noise power} = \frac{\delta^2}{12} ; \text{ For linear quantization.} \\ &\text{or Quantization error (in terms of power)} \end{aligned}$$

... (6.1.23)

Derivation of maximum signal to quantization noise ratio for linear quantization :

From equation 6.1.17 signal to quantization noise ratio is given as,

$$\begin{aligned} \frac{S}{N} &= \frac{\text{Normalized signal power}}{\text{Normalized noise power}} \\ &= \frac{\text{Normalized signal power}}{(\delta^2 / 12)} \end{aligned}$$

... (6.1.24)

From equation 6.1.3 number of bits ' v ' and quantization levels are related as,

$$q = 2^v$$

... (6.1.25)

Putting this value in equation 6.1.10 we have,

$$\delta = \frac{2 x_{\max}}{2^v}$$

... (6.1.26)

Putting this value in equation 6.1.24 we get,

$$\frac{S}{N} = \frac{\text{Normalized signal power}}{\left(\frac{2 x_{\max}}{2^v} \right)^2 / 12}$$

Let normalized signal power be denoted as 'P'.

$$\frac{S}{N} = \frac{P}{\frac{4 x_{\max}^2}{2^{2v}} \times \frac{1}{12}} = \frac{3P}{x_{\max}^2} \cdot 2^{2v}$$

This is the required relation for maximum signal to quantization noise ratio. Thus,

Maximum signal to quantization noise ratio : $\frac{S}{N} = \frac{3P}{x_{\max}^2} \cdot 2^{2v}$

 ... (6.1.27)

This equation shows that signal to noise power ratio of quantizer increases exponentially with increasing bits per sample.

If we assume that input $x(t)$ is normalized, i.e.,

$$x_{\max} = 1 \quad \dots (6.1.28)$$

Then signal to quantization noise ratio will be,

$$\frac{S}{N} = 3 \times 2^{2v} \times P \quad \dots (6.1.29)$$

$$P \leq 1 \quad \dots (6.1.30)$$

Then the signal to noise ratio is given as,

$$\frac{S}{N} \leq 3 \times 2^{2v} \quad \dots (6.1.31)$$

Since $x_{\max} = 1$ and $P \leq 1$, the signal to noise ratio given by above equation is normalized.

Expressing the signal to noise ratio in decibels,

$$\begin{aligned} \left(\frac{S}{N} \right) dB &= 10 \log_{10} \left(\frac{S}{N} \right) dB \quad \text{since power ratio.} \\ &\leq 10 \log_{10} [3 \times 2^{2v}] \\ &\leq (4.8 + 6v) dB \end{aligned}$$

Thus,

Signal to Quantization noise ratio for normalized values of power 'P' and amplitude of input $x(t)$	$:\left(\frac{S}{N} \right) dB \leq (4.8 + 6v) dB$	$\dots (6.1.32)$
--	---	------------------

continue

if the destination signal power 'P' is normalized, i.e. $P \leq 1$

Then the signal to noise ratio is given as,

$$S/N \leq 3 \times 2^{2v}$$

Since $x_{\max} = 1$ and $P \leq 1$, the signal to noise ratio given by above equation is normalized.

Expressing the signal to noise ratio in decibels,

$$S/N \text{ dB} = 10 \log_{10} [S/N] \text{ dB} \quad \text{since power ratio}$$

$$\leq 10 \log_{10} [3 \times 2^{2v}] \text{ dB}$$

$$\leq 10(\log_{10} 3 + 2v \log_{10} 2)$$

$$10(0.48 + 2v (0.30)) \leq$$

$$\leq (4.8 + 6v)$$

$$\leq (4.8 + 6v) \text{ dB}$$

**which is signal to noise ratio for normalized values
power P and amplitude of input $x(t)$**

Signal to noise Ratio in PCM - Equation

Case	Normal equation	dB Equation
General	$\frac{3P}{X_{\max}^2} 2^{2v}$	
Normalized values of power P and amplitude of input x(t)	$\frac{S}{N} \leq 3x 2^{2v}$	$\leq (4.8 + 6v) \text{ dB}$
Sinusoidal input signal	$1.5 x 2^{2v}$	$= 1.8 + 6v \text{ dB}$