

مكتبة المأمون الجامعة

# تحليلات هندسية

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# Lecture No.1

## Laplace Transforms

The Laplace transform is used for solving differential and integral equations. In physics and engineering it is used for analysis of linear time-invariant systems such as electrical circuits, harmonic oscillators, optical devices, and mechanical systems. In such analysis, the Laplace transform is often interpreted as a transformation from the time-domain, in which inputs are functions of complex angular frequency, in radians per unit time. The Laplace transform is that we can convert a whole ODE initial value problem into a Laplace transformed version as functions of  $s$ , simplify the algebra, find the transformed solution  $\tilde{f}(s)$ , then undo the transform to get back to the required solution  $f$  as a function of  $t$ .

### 1-1. Definition and Methods

Given a function  $f(t)$ ,  $t \geq 0$ , its Laplace transform:

$$\begin{aligned} F(s) &= \mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt \\ &= \lim_{A \rightarrow \infty} \int_0^A e^{-st} f(t) dt, \text{ for } s > 0 \end{aligned}$$

We say the transform converges if the limit exists, and diverges if not.

There are three methods to find  $\tilde{f}(s)$  for a given  $f(t)$ :

A - From the definition, e.g.,

1 -  $f(t) = 1$ , for  $t > 0$

Solution:

(1)



$$\text{For } f(t) = 1, \quad \mathcal{L}(1) = \int_0^{\infty} e^{-st}(1) dt$$

$$= \left[ -\frac{1}{s} e^{-st} \right]_0^{\infty}$$

$$= \frac{1}{s} \quad \dots \text{Ans}$$

2-  $f(t) = 3$

Solution!

$$F(s) = \mathcal{L}(f(t)) = \mathcal{L}(3) = \int_0^{\infty} e^{-st}(3) dt$$

$$= \frac{3}{s} \quad \dots \text{Ans}$$

In general if  $f(t) = a$

$$\therefore \boxed{F(s) = \frac{a}{s}}$$

3- For  $f(t) = t$ ,  $\mathcal{L}(t) = \int_0^{\infty} e^{-st} t dt$

$$= \left[ -\frac{1}{s} e^{-st} t \right]_0^{\infty} + \int_0^{\infty} \frac{1}{s} e^{-st} dt$$

$$= \frac{1}{s^2} \quad \dots \text{Ans}$$

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Note! This ans. is by using integration by parts.

4- For  $f(t) = \frac{dy}{dt}$

$$\mathcal{L}\left[\frac{dy}{dt}\right] = \int_0^{\infty} e^{-st} \frac{dy}{dt} dt = \left[ e^{-st} y \right]_0^{\infty}$$

$$= \left[ e^{-st} y \right]_0^{\infty} + \int_0^{\infty} s e^{-st} y dt = -y(0) + s \bar{y}(s)$$

$$\dots \text{Ans}$$

5- For  $f(t) = e^{at}$ ,  $a$  constant

$$\mathcal{L}(e^{at}) = \int_0^{\infty} e^{-st} e^{at} dt$$

$$= \int_0^{\infty} e^{-(s-a)t} dt = \left[ -\frac{1}{s-a} e^{-(s-a)t} \right]_0^{\infty}$$

$$= \frac{1}{s-a}, \quad s > a$$

B. From a property: There are a number of powerful theorems about the properties of transforms; for example;

$$1. \mathcal{L}(af + bg) = a\mathcal{L}(f) + b\mathcal{L}(g)$$

$$2. \mathcal{L}(3t + 4) = 3 \frac{1}{s^2} + 4 \frac{1}{s}$$

$$3. f(t) = u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases} \quad \text{unit step function}$$

Solution:

$$\begin{aligned} \mathcal{L}\{u(t)\} &= \int_0^{\infty} e^{-st} (1) dt + \int_{-\infty}^0 e^{-st} (0) dt \\ &= \int_0^{\infty} e^{-st} dt = \frac{1}{s} \end{aligned}$$

$$4. f(t) = t$$

Solution:

$$\mathcal{L}[f(t)] = \int_0^{\infty} e^{-st} f(t) dt = \int_0^{\infty} e^{-st} t dt$$

using integration by parts

$$\int u dv = uv - \int v du$$

$$u = t \Rightarrow du = dt$$

$$dv = e^{-st} dt \Rightarrow v = -\frac{1}{s} e^{-st}$$

$$\begin{aligned} \therefore \int_0^{\infty} e^{-st} t dt &= -\left[ \frac{t}{s} e^{-st} \right]_0^{\infty} - \int_0^{\infty} -\frac{1}{s} e^{-st} dt \\ &= -\left[ \frac{1}{s^2} \right]_0^{\infty} = \frac{1}{s^2} \end{aligned}$$

In general:

$$\boxed{\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}} \quad n \geq 1$$

$$5. \quad f(t) = t^3$$

Solution:

$$\begin{aligned} \mathcal{L}(t^3) &= \frac{3!}{s^{n+1}} = \frac{3!}{s^{3+1}} = \frac{1 \times 2 \times 3}{s^{3+1}} \\ &= \frac{6}{s^4} \end{aligned}$$

6. The Laplace trans. of  $\sin(at)$  and  $\cos(at)$ ; we can find  $\mathcal{L}(\sin at)$  &  $\mathcal{L}(\cos at)$  using integration by parts twice (i.e.  $u dv = uv - \int v du$ ) or by Euler's formula (De Moivre)

$$\boxed{e^{iat} = \cos at + i \sin at}$$

$$\mathcal{L}[e^{iat}] = \mathcal{L}[\cos at] + \mathcal{L}[i \sin at]$$

$$\begin{aligned} \mathcal{L}[e^{iat}] &= \frac{1}{s - ia} = \frac{1(s + ia)}{(s - ia)(s + ia)} \\ &= \frac{s + ia}{s^2 + a^2} = \frac{s}{s^2 + a^2} + i \frac{a}{s^2 + a^2} \end{aligned}$$

Comparing the real and imaginary part, we get

$$\mathcal{L}[\cos at] = \frac{s}{s^2 + a^2}, \quad \mathcal{L}[\sin at] = \frac{a}{s^2 + a^2}$$

for  $(s > 0)$

Notes:- Relation between Exponential and Trigonometric function are:

$$e^{i\theta} = \cos \theta + i \sin \theta, \quad e^{-i\theta} = \cos \theta - i \sin \theta$$

$$\sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

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$$\bullet \tan \theta = \frac{e^{i\theta} - e^{-i\theta}}{i(e^{i\theta} + e^{-i\theta})} = -i \left( \frac{e^{i\theta} - e^{-i\theta}}{e^{i\theta} + e^{-i\theta}} \right)$$

Hyperbolic Function

$$\bullet \sinh x = \frac{e^x - e^{-x}}{2}$$

$$\bullet \cosh x = \frac{e^x + e^{-x}}{2}$$

$$\bullet \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

7- prove;  $\mathcal{L}(\cosh at) = \frac{s^2}{s^2 - a^2}$   
 solution:

$$\begin{aligned} \mathcal{L}(\cosh at) &= \mathcal{L}\left[\frac{e^{at} + e^{-at}}{2}\right] \\ &= \frac{1}{2} \mathcal{L}(e^{at}) + \frac{1}{2} \mathcal{L}(e^{-at}) \\ &= \frac{1}{2} \left[ \frac{1}{s-a} + \frac{1}{s+a} \right] \\ &= \frac{1}{2} \left[ \frac{s+a+s-a}{s^2 - a^2} \right] = \frac{s}{s^2 - a^2} \quad \text{Ans.} \end{aligned}$$

8- Prove:  $\mathcal{L}(\sinh at) = \frac{a}{s^2 - a^2}$

$$\mathcal{L}(\sinh at) = \mathcal{L}\left[\frac{1}{2}(e^{at} - e^{-at})\right]$$

$$= \frac{1}{2} [\mathcal{L}(e^{at}) - \mathcal{L}(e^{-at})]$$

$$= \frac{1}{2} \left[ \frac{1}{s-a} - \frac{1}{s+a} \right]$$

$$= \frac{1}{2} \left[ \frac{s+a-s+a}{s^2 + a^2} \right]$$

$$= \frac{a}{s^2 + a^2} \quad \text{Ans}$$

9- prove;  $\mathcal{L}(\sin at) = \frac{a}{s^2 + a^2}$

$$\begin{aligned}
 \mathcal{L}(\sin at) &= \mathcal{L}\left[\frac{e^{iat} - e^{-iat}}{2i}\right] = \frac{1}{2i} [\mathcal{L}(e^{iat}) - \mathcal{L}(e^{-iat})] \\
 &= \frac{1}{2i} [\mathcal{L}(e^{iat}) - \mathcal{L}(e^{-iat})] \\
 &= \frac{1}{2i} \left[\frac{1}{s-ia} - \frac{1}{s+ia}\right] = \frac{1}{2i} \left[\frac{s+ia-s+ia}{s^2+a^2}\right] \\
 &= \frac{1}{2i} \left[\frac{2ia}{s^2+a^2}\right] = \frac{a}{s^2+a^2} \dots \text{Ans.}
 \end{aligned}$$

10- prove;  $\mathcal{L}(\cos at) = \frac{s}{s^2+a^2}$

Solution:  $\mathcal{L}(\cos at) = \mathcal{L}\left[\frac{e^{iaa} + e^{-iaa}}{2}\right]$

$$\begin{aligned}
 \mathcal{L}(\cos at) &= \frac{1}{2} [\mathcal{L}(e^{iaa} + e^{-iaa})] = \frac{1}{2} [\mathcal{L}(e^{iaa}) + \mathcal{L}(e^{-iaa})] \\
 &= \frac{1}{2} \left[\frac{1}{s-ia} + \frac{1}{s+ia}\right] = \frac{1}{2} \left[\frac{s+ia+s-ia}{s^2+a^2}\right] \\
 &= \frac{1}{2} \cdot \frac{2s}{s^2+a^2} = \frac{s}{s^2+a^2} \dots \text{Ans.}
 \end{aligned}$$

11. From the first principle, find the Laplace transform of  $(1 + \cos 2t)$ .

Solution:

$$\begin{aligned}
 \mathcal{L}(1 + \cos 2t) &= \int_0^{\infty} e^{-st} (1 + \cos 2t) dt \\
 &= \int_0^{\infty} e^{-st} \left[1 + \frac{e^{2it} + e^{-2it}}{2}\right] dt \\
 &= \frac{1}{2} \int_0^{\infty} \left[2e^{-st} + e^{(-s+2i)t} + e^{(-s-2i)t}\right] dt \\
 &= \frac{1}{2} \left[\frac{2e^{-st}}{-s} + \frac{e^{(-s+2i)t}}{-s+2i} + \frac{e^{(-s-2i)t}}{-s-2i}\right]_0^{\infty} \\
 &= \frac{1}{2} \left[\left(0 + \frac{2}{s}\right) + \frac{1}{-s+2i} (0-1) + \frac{1}{-s-2i} (0-1)\right] \\
 &= \frac{1}{2} \left[\frac{2}{s} + \frac{1}{s-2i} + \frac{1}{s+2i}\right] = \frac{1}{2} \left[\frac{2}{s} + \frac{2s}{s^2+4}\right]
 \end{aligned}$$

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$$= \frac{1}{s} + \frac{s}{s^2+4} = \frac{2s^2+4}{s(s^2+4)} \quad \dots \text{Ans.}$$

12. Find the laplace transform of  $f(t)$  defined as:

$$f(t) = \frac{t}{k}, \text{ when } 0 < t < k$$

$$= 1, \text{ when } t > k$$

Solution:

$$\begin{aligned} F(s) = \mathcal{L}[f(t)] &= \int_0^k \frac{t}{k} e^{-st} dt + \int_k^\infty 1 \cdot e^{-st} dt \\ &= \frac{1}{k} \left\{ \left[ t \frac{e^{-st}}{-s} \right]_0^k - \int_0^k \frac{e^{-st}}{-s} dt \right\} + \left[ \frac{e^{-st}}{-s} \right]_k^\infty \\ &= \frac{1}{k} \left[ \frac{k e^{-ks}}{-s} - \left[ \frac{e^{-st}}{s^2} \right]_0^k \right] + \frac{e^{-ks}}{s} \\ &= \frac{1}{k} \left[ \frac{k e^{-ks}}{-s} - \frac{e^{-sk}}{s^2} + \frac{1}{s^2} \right] + \frac{e^{-ks}}{s} \\ &= -\frac{e^{-sk}}{s} - \frac{1}{k} \frac{e^{-ks}}{s^2} + \frac{1}{k} \frac{1}{s^2} + \frac{e^{-ks}}{s} \\ &= \frac{1}{ks^2} [-e^{-ks} + 1] \quad \dots \text{Ans.} \end{aligned}$$

13. Find the laplace transform of

$$f(t) = \begin{cases} 1, & 0 \leq t < 2, \\ t-2, & 2 \leq t \end{cases}$$

Solution:

$$\begin{aligned} F(s) = \mathcal{L}[f(t)] &= \int_0^\infty e^{-st} f(t) dt = \int_0^2 e^{-st} (1) dt + \int_2^\infty (t-2) e^{-st} dt \\ F(s) &= \left[ \frac{1}{-s} e^{-st} \right]_{t=0}^2 + (t-2) \left[ \frac{1}{-s} e^{-st} \right]_{t=2}^\infty - \int_2^\infty \frac{1}{-s} e^{-st} dt \\ &= \frac{1}{-s} (e^{-2s} - 1) + (0-0) + \frac{1}{s} \cdot \frac{1}{-s} e^{-st} \Big|_{t=2}^\infty \\ &= \frac{1}{-s} (e^{-2s} - 1) + \frac{1}{s^2} e^{-2s} \quad \dots \text{Ans.} \end{aligned}$$

C. From a list; Computer algebra packages like Mathematica, Matlab and Maple know Laplace Transforms of all the functions you are likely to encounter, so you have access to these online, and the packages have also an inversion routine to find a function  $f$  from a given  $F$ .

## 1-2. Properties of Laplace Transform:

In the following list of theorems we assume, unless otherwise stated, that all functions satisfy the condition (if  $F(t)$  is sectionally continuous in every finite interval  $0 \leq t \leq N$  and of exponential order  $\gamma$  for  $t > N$ , then its Laplace transform  $f(s)$  exists for all  $s > \gamma$ .) so that their Laplace transforms exist.

1. If  $a$  and  $b$  are constant while  $f, g$  function for which Laplace Transform exists then:-

$$\mathcal{L}[af(t) + bg(t)] = aF(s) + bG(s)$$

[This is called Linearity operator Or Linearity property]

example (1):

$$\begin{aligned}\mathcal{L}(4t^2 - 3\cos 2t + 5e^{-t}) &= 4\mathcal{L}(t^2) - 3\mathcal{L}(\cos 2t) + 5\mathcal{L}(e^{-t}) \\ &= 4\left(\frac{2!}{s^3}\right) - 3\left(\frac{s}{s^2+4}\right) + 5\left(\frac{1}{s+1}\right) \\ &= \frac{8}{s^3} - \frac{3s}{s^2+4} + \frac{5}{s+1}\end{aligned}$$

2. First translation or shifting property:

$$\mathcal{L}[e^{\pm at} f(t)] = F(s \mp a)$$

example (2): prove  $\mathcal{L}[e^{at} f(t)] = F(s-a)$

Solution;

$$\begin{aligned} \mathcal{L}[e^{at} f(t)] &= \int_0^{\infty} e^{-st} \cdot e^{at} f(t) dt = \int_0^{\infty} e^{-(s-a)t} f(t) dt \\ &= \int_0^{\infty} e^{-rt} f(t) dt \quad \text{where } r = s - a \end{aligned}$$

$$= F(r) = F(s - a)$$

With the help of this property, we can have the following important results:

$$(1) \mathcal{L}(e^{at} t^n) = \frac{n!}{(s-a)^{n+1}} \quad \mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$$

$$(2) \mathcal{L}(e^{at} \cosh bt) = \frac{s-a}{(s-a)^2 - b^2}$$

$$(3) \mathcal{L}(e^{at} \sinh bt) = \frac{b}{(s-a)^2 - b^2}$$

$$(4) \mathcal{L}(e^{at} \sin bt) = \frac{b}{(s-a)^2 + b^2}$$

$$(5) \mathcal{L}(e^{at} \cos bt) = \frac{s-a}{(s-a)^2 + b^2}$$

Example (3): Find the Laplace transform of  $\cos^2 t$ .

Notes a -  $\sin 2A = 2 \sin A \cos A$

b -  $\cos 2A = \cos^2 A - \sin^2 A$

$= 1 - 2 \sin^2 A$

$= 2 \cos^2 A - 1$

c -  $\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$

Double Angle Formulas

Solution:  $\cos 2t = 2 \cos^2 t - 1$

$\cos^2 t = \frac{1}{2} [\cos 2t + 1]$

$\mathcal{L}(\cos^2 t) = \mathcal{L}\left[\frac{1}{2} (\cos 2t + 1)\right]$

$= \frac{1}{2} \left[ \frac{s}{s^2 + 2^2} + \frac{1}{s} \right]$

$= \frac{1}{2} \left[ \frac{s}{s^2 + 4} + \frac{1}{s} \right] \therefore \text{Ans}$

Example (4): Find the Laplace of  $t^{-\frac{1}{2}}$

Solution:  $L(t^n) = \frac{\sqrt{n+1}}{s^{n+1}}$  as given above  
here;  $n = -\frac{1}{2}$

$$\therefore L(t^{-\frac{1}{2}}) = \frac{\sqrt{-\frac{1}{2}+1}}{s^{-\frac{1}{2}+1}} = \frac{\sqrt{\frac{1}{2}}}{\sqrt{s}} \\ = \sqrt{\frac{1/2}{s}} \quad \dots \text{Ans.}$$

Example (5): Find the Laplace Tran. of  $t^2 \sin at$ .

Solution:

$$L(t^2 \sin at) = L\left[t^2 \frac{e^{iat} - e^{-iat}}{2i}\right] \\ = \frac{1}{2i} [L(t^2 e^{iat}) - L(t^2 e^{-iat})] \\ = \frac{1}{2i} \left[ \frac{1}{(s-ia)^2} - \frac{1}{(s+ia)^2} \right] \\ = \frac{1}{2i} \left[ \frac{(s+ia)^2 - (s-ia)^2}{(s-ia)^2 (s+ia)^2} \right] \\ = \frac{1}{2i} \frac{(s^2 + 2ias - s^2) - (s^2 - 2ias - s^2)}{(s^2 + a^2)^2} \\ = \frac{1}{2i} \frac{4ias}{(s^2 + a^2)^2} = \frac{2as}{(s^2 + a^2)^2} \quad \dots \text{Ans.}$$

Example (6): Find the Laplace Tran of the following

a-  $L[t^3 + 5t - 2] = L(t^3) + 5L(t) - 2L(1)$

$$= \frac{3!}{s^4} + 5 \cdot \frac{1}{s^2} - 2 \cdot \frac{1}{s} \quad \dots \text{Ans.}$$

b-  $L[e^{2t}(t^3 + 5t - 2)] = \frac{3!}{(s-2)^4} + 5 \cdot \frac{1}{(s-2)^2} - 2 \cdot \frac{1}{s-2}$

c-  $L[(t^2 + 4)e^{2t} - e^t \cos t] = \frac{2}{(s-2)^3} + \frac{4}{s-2} - \frac{s+1}{(s+1)^2 + 1}$

because;  $L[t^2 + 4] = \frac{2}{s^3} + \frac{4}{s} \Rightarrow L[(t^2 + 4)e^{2t}] = \frac{2}{(s-2)^3} + \frac{4}{s-2}$

3. Laplace Trans. of the Derivative of  $f(t)$   
 $\mathcal{L}[f'(t)] = s \mathcal{L}[f(t)] - f(0)$

where  $\mathcal{L}[f(t)] = F(s)$

prove:

$$\begin{aligned}\mathcal{L}[f'(t)] &= \int_0^{\infty} e^{-st} f'(t) dt \\ &= \left[ e^{-st} f(t) \right]_0^{\infty} - \int_0^{\infty} (-s e^{-st}) f(t) dt \\ &= -f(\infty) + s \int_0^{\infty} e^{-st} f(t) dt, \quad [e^{-st} f(t)]_0^{\infty} = 0 \text{ when } t = \infty \\ &= -f(0) + s \mathcal{L}[f(t)] \\ &= s \mathcal{L}[f(t)] - f(0) \quad \text{Ans.}\end{aligned}$$

4. Laplace Trans. of Derivative of order  $n$ .

$$\mathcal{L}[f^{(n)}(t)] = s^n \mathcal{L}[f(t)] - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) - \dots - f^{(n-1)}(0)$$

5. Laplace Trans. of Integral of  $f(t)$

$$\mathcal{L}\left[\int_0^t f(t) dt\right] = \frac{1}{s} F(s)$$

6. Time scaling property:

$$\mathcal{L}[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right)$$

Example 7. Find  $\mathcal{L}[t^4 - 4t + 5 + 3\sin 2t]$

Solution;

$$= \mathcal{L}[t^4] - \mathcal{L}(4t) + \mathcal{L}5 + \mathcal{L}(3\sin 2t)$$

$$= \frac{24}{s^5} - \frac{4}{s^2} + \frac{5}{s} + \frac{6}{s^2 + 4}$$

$$\left\{ \begin{aligned} \mathcal{L}[\sin(at)] &= \frac{a}{s^2 + a^2} \\ a &= 2 \\ \mathcal{L}(3\sin 2t) &= \frac{3 \times 2}{s^2 + (2)^2} \end{aligned} \right.$$

7 Laplace Trans. of  $t \cdot f(t)$  (Multiplication by  $t$ )  
 if  $\mathcal{L}[f(t)] = F(s)$ , then

$$\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} [F(s)]$$

Also:

$$a. \mathcal{L}[t^2 f(t)] = (-1)^2 \frac{d^2}{ds^2} [F(s)]$$

$$b. \mathcal{L}[t^3 f(t)] = (-1)^3 \frac{d^3}{ds^3} [F(s)]$$

$$c. \mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} [F(s)]$$

Example 8: Find the Laplace trans. of  $t \sinh at$   
 solution:  $\mathcal{L}(\sinh at) = \frac{a}{s^2 - a^2}$

$$\mathcal{L}[t \sinh at] = -\frac{d}{ds} \left[ \frac{a}{s^2 - a^2} \right]$$

Note!

$$\frac{d}{dx} \left( \frac{u}{v} \right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$$

$$\mathcal{L}(t \sinh at) = \frac{2as}{(s^2 - a^2)^2} \dots \text{Ans}$$

Example 9: Find the Laplace trans. of  $t^2 \cos at$ .

$$\text{solution: } \mathcal{L}(\cos at) = \frac{s}{s^2 + a^2}$$

$$\mathcal{L}(t^2 \cos at) = (-1)^2 \frac{d^2}{ds^2} \left[ \frac{s}{s^2 + a^2} \right] = \frac{d}{ds} \frac{(s^2 + a^2) \cdot 1 - s(2s)}{(s^2 + a^2)^2}$$

$$= \frac{d}{ds} \left[ \frac{a^2 - s^2}{(s^2 + a^2)^2} \right]$$

$$= \frac{(s^2 + a^2)^2 (-2s) - (a^2 - s^2) \cdot 2(s^2 + a^2)(2s)}{(s^2 + a^2)^4}$$

$$= \frac{-2s^2 - 2a^2s - 4a^2s + 4s^3}{(s^2 + a^2)^3}$$

$$= \frac{2s(s^2 - 3a^2)}{(s^2 + a^2)^3} \dots \text{Ans.}$$

8. Laplace Trans. of  $\frac{1}{t} f(t)$  [Division by  $t$ ]

if  $\mathcal{L}[f(t)] = F(s)$ , then  $\mathcal{L}\left[\frac{1}{t} f(t)\right] = \int_s^\infty F(s) ds$

prove.  $\mathcal{L}[f(t)] = F(s)$  or  $F(s) = \int_0^\infty e^{-st} f(t) dt$  ... (1)

Integrating (1) w.r.t 's', we have

$$\begin{aligned} \int_s^\infty F(s) ds &= \int_s^\infty \left[ \int_0^\infty e^{-st} f(t) dt \right] ds \\ &= \int_0^\infty \left[ \int_s^\infty e^{-st} f(t) ds \right] dt = \int_0^\infty \left[ \frac{e^{-st}}{-t} \right]_s^\infty dt \\ &= \int_0^\infty \frac{-f(t)}{t} [e^{-st}]_s^\infty dt = \int_0^\infty \frac{-f(t)}{t} [0 - e^{-st}] dt \\ &= \int_0^\infty e^{-st} \left[ \frac{1}{t} \cdot f(t) \right] dt = \mathcal{L}\left[\frac{1}{t} f(t)\right] \end{aligned}$$

$$\mathcal{L}\left[\frac{1}{t} f(t)\right] = \int_s^\infty F(s) ds$$

Example 10; Find the Laplace trans. of  $\frac{\sin 2t}{t}$

Solution.  $\mathcal{L}(\sin 2t) = \frac{2}{s^2 + 4}$

$$\begin{aligned} \mathcal{L}\left[\frac{\sin 2t}{t}\right] &= \int_s^\infty \frac{2}{s^2 + 4} ds = 2 \cdot \frac{1}{2} \left[ \tan^{-1} \frac{s}{2} \right]_s^\infty \\ &= \left[ \tan^{-1} \infty - \tan^{-1} \frac{s}{2} \right] = \frac{\pi}{2} - \tan^{-1} \frac{s}{2} \\ &= \cot^{-1} \frac{s}{2} \quad \dots \text{Ans.} \end{aligned}$$

Example 11; Find the Laplace trans. of  $f(t) = \int_0^t \frac{\sin t}{t} dt$

Solution:

$$\begin{aligned} \mathcal{L} \sin t &= \frac{1}{s^2 + 1} \\ \mathcal{L} \frac{\sin t}{t} &= \int_s^\infty \frac{1}{s^2 + 1} ds = \left[ \tan^{-1} s \right]_s^\infty = \frac{\pi}{2} - \tan^{-1} s \\ \mathcal{L} \int_0^t \frac{\sin t}{t} dt &= \frac{1}{s} \left[ \frac{\pi}{2} - \tan^{-1} s \right] \dots \text{Ans.} \end{aligned}$$

Note

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## TRIGONOMETRIC FUNCTIONS

### ADDITION FORMULAS

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\cot(A \pm B) = \frac{\cot A \cot B \mp 1}{\cot B \pm \cot A}$$

### DOUBLE ANGLE FORMULAS

$$\sin 2A = 2 \sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A = 1 - 2 \sin^2 A = 2 \cos^2 A - 1$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

### HALF ANGLE FORMULAS

$$\sin \frac{A}{2} = \pm \sqrt{\frac{1 - \cos A}{2}} \quad \left[ \begin{array}{l} + \text{ if } A/2 \text{ is in quadrant I or II} \\ - \text{ if } A/2 \text{ is in quadrant III or IV} \end{array} \right]$$

$$\cos \frac{A}{2} = \pm \sqrt{\frac{1 + \cos A}{2}} \quad \left[ \begin{array}{l} + \text{ if } A/2 \text{ is in quadrant I or IV} \\ - \text{ if } A/2 \text{ is in quadrant II or III} \end{array} \right]$$

$$\tan \frac{A}{2} = \pm \sqrt{\frac{1 - \cos A}{1 + \cos A}} \quad \left[ \begin{array}{l} + \text{ if } A/2 \text{ is in quadrant I or III} \\ - \text{ if } A/2 \text{ is in quadrant II or IV} \end{array} \right]$$

$$= \frac{\sin A}{1 + \cos A} = \frac{1 - \cos A}{\sin A} = \csc A - \cot A$$

### MULTIPLE ANGLE FORMULAS

$$\sin 3A = 3 \sin A - 4 \sin^3 A$$

$$\cos 3A = 4 \cos^3 A - 3 \cos A$$

$$\tan 3A = \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}$$

$$\sin 4A = 4 \sin A \cos A - 8 \sin^3 A \cos A$$

$$\cos 4A = 8 \cos^4 A - 8 \cos^2 A + 1$$

$$\tan 4A = \frac{4 \tan A - 4 \tan^3 A}{1 - 6 \tan^2 A + \tan^4 A}$$

$$\sin 5A = 5 \sin A - 20 \sin^3 A + 16 \sin^5 A$$

$$\cos 5A = 16 \cos^5 A - 20 \cos^3 A + 5 \cos A$$

$$\tan 5A = \frac{\tan^5 A - 10 \tan^3 A + 5 \tan A}{1 - 10 \tan^2 A + 5 \tan^4 A}$$

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## POWERS OF TRIGONOMETRIC FUNCTIONS

$$5.53 \quad \sin^2 A = \frac{1}{2} - \frac{1}{2} \cos 2A$$

$$5.54 \quad \cos^2 A = \frac{1}{2} + \frac{1}{2} \cos 2A$$

$$5.55 \quad \sin^3 A = \frac{3}{4} \sin A - \frac{1}{4} \sin 3A$$

$$5.56 \quad \cos^3 A = \frac{3}{4} \cos A + \frac{1}{4} \cos 3A$$

$$5.57 \quad \sin^4 A = \frac{3}{8} - \frac{1}{2} \cos 2A + \frac{1}{8} \cos 4A$$

$$5.58 \quad \cos^4 A = \frac{3}{8} + \frac{1}{2} \cos 2A + \frac{1}{8} \cos 4A$$

$$5.59 \quad \sin^5 A = \frac{5}{8} \sin A - \frac{5}{16} \sin 3A + \frac{1}{16} \sin 5A$$

$$5.60 \quad \cos^5 A = \frac{5}{8} \cos A + \frac{5}{16} \cos 3A + \frac{1}{16} \cos 5A$$

See also formulas 5.70 through 5.73.

## SUM, DIFFERENCE AND PRODUCT OF TRIGONOMETRIC FUNCTIONS

$$\sin A + \sin B = 2 \sin \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B)$$

$$\sin A - \sin B = 2 \cos \frac{1}{2}(A+B) \sin \frac{1}{2}(A-B)$$

$$\cos A + \cos B = 2 \cos \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B)$$

$$\cos A - \cos B = 2 \sin \frac{1}{2}(A+B) \sin \frac{1}{2}(B-A)$$

$$\sin A \sin B = \frac{1}{2} \{ \cos(A-B) - \cos(A+B) \}$$

$$\cos A \cos B = \frac{1}{2} \{ \cos(A-B) + \cos(A+B) \}$$

$$\sin A \cos B = \frac{1}{2} \{ \sin(A-B) + \sin(A+B) \}$$

## RELATIONSHIP BETWEEN EXPONENTIAL AND TRIGONOMETRIC FUNCTIONS

$$e^{i\theta} = \cos \theta + i \sin \theta, \quad e^{-i\theta} = \cos \theta - i \sin \theta$$

These are called *Euler's identities*. Here  $i$  is the imaginary unit [see page 21].

$$\sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\tan \theta = \frac{e^{i\theta} - e^{-i\theta}}{i(e^{i\theta} + e^{-i\theta})} = -i \left( \frac{e^{i\theta} - e^{-i\theta}}{e^{i\theta} + e^{-i\theta}} \right)$$

$$\cot \theta = i \left( \frac{e^{i\theta} + e^{-i\theta}}{e^{i\theta} - e^{-i\theta}} \right)$$

$$\sec \theta = \frac{2}{e^{i\theta} + e^{-i\theta}}$$

$$\csc \theta = \frac{2i}{e^{i\theta} - e^{-i\theta}}$$

# HYPERBOLIC FUNCTIONS

## DEFINITION OF HYPERBOLIC FUNCTIONS

$$\text{Hyperbolic sine of } x = \sinh x = \frac{e^x - e^{-x}}{2}$$

$$\text{Hyperbolic cosine of } x = \cosh x = \frac{e^x + e^{-x}}{2}$$

$$\text{Hyperbolic tangent of } x = \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\text{Hyperbolic cotangent of } x = \coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

$$\text{Hyperbolic secant of } x = \operatorname{sech} x = \frac{2}{e^x + e^{-x}}$$

$$\text{Hyperbolic cosecant of } x = \operatorname{csch} x = \frac{2}{e^x - e^{-x}}$$

## RELATIONSHIPS AMONG HYPERBOLIC FUNCTIONS

$$\tanh x = \frac{\sinh x}{\cosh x}$$

$$\coth x = \frac{1}{\tanh x} = \frac{\cosh x}{\sinh x}$$

$$\operatorname{sech} x = \frac{1}{\cosh x}$$

$$\operatorname{csch} x = \frac{1}{\sinh x}$$

$$\cosh^2 x - \sinh^2 x = 1$$

$$\operatorname{sech}^2 x + \tanh^2 x = 1$$

$$\coth^2 x - \operatorname{csch}^2 x = 1$$

## FUNCTIONS OF NEGATIVE ARGUMENTS

$$\sinh(-x) = -\sinh x$$

$$8.15 \quad \cosh(-x) = \cosh x$$

$$8.16 \quad \tanh(-x) = -\tanh x$$

$$\operatorname{csch}(-x) = -\operatorname{csch} x$$

$$8.18 \quad \operatorname{sech}(-x) = \operatorname{sech} x$$

$$8.19 \quad \coth(-x) = -\coth x$$

### ADDITION FORMULAS

$$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$$

$$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$$

$$\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$$

$$\coth(x \pm y) = \frac{\coth x \coth y \pm 1}{\coth y \pm \coth x}$$

### DOUBLE ANGLE FORMULAS

$$\sinh 2x = 2 \sinh x \cosh x$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x = 2 \cosh^2 x - 1 = 1 + 2 \sinh^2 x$$

$$\tanh 2x = \frac{2 \tanh x}{1 + \tanh^2 x}$$

### HALF ANGLE FORMULAS

$$\sinh \frac{x}{2} = \pm \sqrt{\frac{\cosh x - 1}{2}} \quad [+ \text{ if } x > 0, - \text{ if } x < 0]$$

$$\cosh \frac{x}{2} = \sqrt{\frac{\cosh x + 1}{2}}$$

$$\begin{aligned} \tanh \frac{x}{2} &= \pm \sqrt{\frac{\cosh x - 1}{\cosh x + 1}} \quad [+ \text{ if } x > 0, - \text{ if } x < 0] \\ &= \frac{\sinh x}{\cosh x + 1} = \frac{\cosh x - 1}{\sinh x} \end{aligned}$$

### MULTIPLE ANGLE FORMULAS

$$\sinh 3x = 3 \sinh x + 4 \sinh^3 x$$

$$\cosh 3x = 4 \cosh^3 x - 3 \cosh x$$

$$\tanh 3x = \frac{3 \tanh x + \tanh^3 x}{1 + 3 \tanh^2 x}$$

$$\sinh 4x = 8 \sinh^3 x \cosh x + 4 \sinh x \cosh^3 x$$

$$\cosh 4x = 8 \cosh^4 x - 8 \cosh^2 x + 1$$

$$\tanh 4x = \frac{4 \tanh x + 4 \tanh^3 x}{1 + 6 \tanh^2 x + \tanh^4 x}$$

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## Lecture NO. 2

### The Inverse Laplace Transform

The inverse laplace transform of the  $F(s)$  is the unique function  $f(t)$  that continuous on  $[0, \infty]$  and satisfies  $\mathcal{L}[f(t)] = F(s)$ , we will use the notation  $[\mathcal{L}^{-1}(F(s))]$  to denote the inverse

laplace transform of  $F(s)$ , and will write as:

$$\mathcal{L}^{-1}[F(s)] = f(t)$$

where  $\mathcal{L}^{-1}$  is called the inverse laplace transform operator

From the application point of view, the inverse laplace transform is very useful.

2-1. Some important formulae:

$$1 - \mathcal{L}^{-1}\left(\frac{1}{s}\right) = 1, \quad 2 - \mathcal{L}^{-1}\left[\frac{n!}{s^{n+1}}\right] = t^n$$

$$3 - \mathcal{L}^{-1}\left(\frac{1}{s-a}\right) = e^{at}, \quad 4 - \mathcal{L}^{-1}\left[\frac{s}{s^2-a^2}\right] = \cosh at$$

$$5 - \mathcal{L}^{-1}\left[\frac{1}{s^2-a^2}\right] = \frac{1}{a} \sinh at, \quad 6 - \mathcal{L}^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a} \sin at$$

$$7 - \mathcal{L}^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at, \quad 8 - \mathcal{L}^{-1}\left[\frac{s-a}{(s-a)^2+b^2}\right] = e^{at} \cos bt$$

$$9 - \mathcal{L}^{-1}\left[\frac{1}{(s-a)^2+b^2}\right] = \frac{1}{b} e^{at} \sin bt, \quad 10 - \mathcal{L}^{-1}\left[\frac{s-a}{(s-a)^2+b^2}\right] = e^{at} \cos bt$$

$$11 - \mathcal{L}^{-1}\left[\frac{1}{(s-a)^2-b^2}\right] = \frac{1}{b} e^{at} \sinh bt, \quad 12 - \mathcal{L}^{-1}\left[\frac{s-a}{(s-a)^2-b^2}\right] = e^{at} \cosh bt$$

$$13 - \mathcal{L}^{-1}\left[\frac{1}{(s^2+b^2)^2}\right] = \frac{1}{2a^3} (\sin at - at \cos at), \quad 14 - \mathcal{L}^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \frac{1}{2a} t \sin at$$

$$15 - \mathcal{L}^{-1}\left[\frac{s^2-a^2}{(s^2+a^2)^2}\right] = t \cos at, \quad 16 - \mathcal{L}^{-1}(1) = \delta(t)$$

$$17 - \mathcal{L}^{-1}\left[\frac{s^2}{(s^2+a^2)^2}\right] = \frac{1}{2a} [\sin at + \cos at]$$

2-2. Some properties of Inverse Laplace Trans.

A. Linearity property.

If  $b$  and  $c$  are any constants while  $F_1(s)$  and

$F_1(s)$  and  $F_2(s)$  are the Laplace transforms of  $f_1(t)$  and  $f_2(t)$  respectively, then

$$\mathcal{L}^{-1}[bF_1(s) + cF_2(s)] = b\mathcal{L}^{-1}(F_1(s)) + c\mathcal{L}^{-1}(F_2(s)) \\ = b f_1(t) + c f_2(t)$$

The result is easily extended to more than two functions.

Example 1-

$$\mathcal{L}^{-1}\left[\frac{4}{s-2} - \frac{3s}{s^2+16} + \frac{5}{s^2+4}\right] = 4\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} - \\ 3\mathcal{L}^{-1}\left\{\frac{s}{s^2+16}\right\} + 5\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\} \\ = 4e^{2t} - 3\cos 4t + \frac{5}{2}\sin 2t \text{ --- Ans.}$$

Example 2

$$\mathcal{L}^{-1}\left[\frac{7}{s^3}\right] = \frac{7}{2}\mathcal{L}^{-1}\left[\frac{2}{s^3}\right] = \frac{7}{2}t^2 \text{ --- Ans.}$$

Example 3

$$\mathcal{L}^{-1}\left[\frac{3s+4}{s^2+4}\right] = \mathcal{L}^{-1}\frac{3s}{s^2+4} + \mathcal{L}^{-1}\frac{4}{s^2+4} = 3\mathcal{L}^{-1}\frac{s}{s^2+4} + 2\mathcal{L}^{-1}\frac{2}{s^2+4} \\ = 3\cos 2t + 2\sin 2t \text{ --- Ans.}$$

Example 4.

$$\mathcal{L}^{-1}\left[\frac{2}{(s+5)^4}\right] = 2\mathcal{L}^{-1}\left[\frac{1}{(s+5)^4}\right] = 2e^{-5t}\mathcal{L}^{-1}\frac{1}{s^4} \\ = \frac{2}{3!}e^{-5t}\mathcal{L}^{-1}\frac{3!}{s^4} = \frac{1}{3}e^{-5t}t^3 \text{ --- Ans.}$$

Example 5.

$$\mathcal{L}^{-1}\left[\frac{6}{2s-3}\right] = \mathcal{L}^{-1}\left[\frac{3}{s-\frac{3}{2}}\right] = 3e^{\frac{3}{2}t}$$

Example 6.

$$\mathcal{L}^{-1}\left[\frac{3+4s}{9s^2-16}\right] = \frac{1}{9}\mathcal{L}^{-1}\left[\frac{3}{s^2-\frac{16}{9}}\right] + \frac{4}{9}\mathcal{L}^{-1}\left[\frac{s}{s^2-\frac{16}{9}}\right]$$

$$= \frac{1}{9} \sinh \frac{4}{3} t + \frac{4}{9} \cosh \frac{4}{3} t \quad \dots \text{Ans}$$

Example 7.

$$\begin{aligned} \mathcal{L}^{-1} \left[ \frac{3s-5}{4(s^2-s+\frac{37}{4})} \right] &= \frac{1}{4} \mathcal{L}^{-1} \left[ \frac{3s-5}{(s^2-s+\frac{1}{4})+9} \right] = \frac{1}{4} \mathcal{L}^{-1} \left[ \frac{3s-5}{(s-\frac{1}{2})^2+3^2} \right] \\ &= \frac{1}{4} \mathcal{L}^{-1} \left[ \frac{3s}{(s-\frac{1}{2})^2+3^2} \right] - \frac{5}{4} \mathcal{L}^{-1} \left[ \frac{1}{(s-\frac{1}{2})^2+3^2} \right] \\ &= \frac{3}{4} \mathcal{L}^{-1} \left[ \frac{(s-\frac{1}{2})+\frac{1}{2}}{(s-\frac{1}{2})^2+9} \right] - \frac{5}{12} \mathcal{L}^{-1} \left[ \frac{3}{(s-\frac{1}{2})^2+3^2} \right] \\ &= \frac{3}{4} \mathcal{L}^{-1} \left[ \frac{s-\frac{1}{2}}{(s-\frac{1}{2})^2+9} + \frac{3}{8(s-\frac{1}{2})^2+9} \right] - \frac{5}{12} e^{\frac{1}{2}t} \sin 3t \\ &= \frac{3}{4} e^{\frac{1}{2}t} \cos 3t + \frac{1}{8} e^{\frac{1}{2}t} \sin 3t - \frac{5}{12} e^{\frac{1}{2}t} \sin 3t \\ &= \frac{3}{4} e^{\frac{1}{2}t} \cos 3t - \frac{7}{24} e^{\frac{1}{2}t} \sin 3t \\ &= \frac{1}{4} e^{\frac{1}{2}t} \left( 3 \cos 3t - \frac{7}{6} \sin 3t \right) = f(t) \quad \dots \text{Ans.} \end{aligned}$$

Example 8.

$$\begin{aligned} \mathcal{L}^{-1} \left[ \frac{3s-5}{4s^2+4s+1} \right] &= \frac{1}{4} \mathcal{L}^{-1} \left[ \frac{3s-5}{s^2+s+\frac{1}{4}} \right] = \frac{1}{4} \mathcal{L}^{-1} \left[ \frac{3s-5}{(s+\frac{1}{2})^2} \right] \\ &= \frac{3}{4} \mathcal{L}^{-1} \left[ \frac{s}{(s+\frac{1}{2})^2} \right] - \frac{5}{4} \mathcal{L}^{-1} \left[ \frac{1}{(s+\frac{1}{2})^2} \right] = \frac{3}{4} \mathcal{L}^{-1} \left[ \frac{(s+\frac{1}{2})-\frac{1}{2}}{(s+\frac{1}{2})^2} \right] - \frac{5}{4} \mathcal{L}^{-1} \left[ \frac{1}{(s+\frac{1}{2})^2} \right] \\ &= \frac{3}{4} \mathcal{L}^{-1} \left[ \frac{(s+\frac{1}{2})}{(s+\frac{1}{2})^2} \right] - \frac{3}{8} \mathcal{L}^{-1} \left[ \frac{1}{(s+\frac{1}{2})^2} \right] - \frac{5}{4} \mathcal{L}^{-1} \left[ \frac{1}{(s+\frac{1}{2})^2} \right] \\ &= \frac{3}{4} \mathcal{L}^{-1} \left[ \frac{1}{s+\frac{1}{2}} \right] - \frac{13}{8} \mathcal{L}^{-1} \left[ \frac{1}{(s+\frac{1}{2})^2} \right] = \frac{3}{4} e^{-\frac{1}{2}t} - \frac{13}{8} e^{-\frac{1}{2}t} t \\ &= \frac{1}{4} e^{-\frac{1}{2}t} \left( 3 - \frac{13}{2} t \right) \quad \dots \text{Ans.} \end{aligned}$$

Example 9.

$$\begin{aligned} \mathcal{L}^{-1} \left[ \frac{3}{s^2+4s+13} \right] &= \mathcal{L}^{-1} \left[ \frac{3}{(s^2+4s+4)-4+13} \right] = \mathcal{L}^{-1} \left[ \frac{3}{(s+2)^2+9} \right] \\ &= e^{-2t} \mathcal{L}^{-1} \left[ \frac{3}{s^2+9} \right] \end{aligned}$$

Example 10. Find the inverse laplace transform of :

a.  $\frac{2s-5}{9s^2-25}$  & b.  $\frac{s-2}{6s^2+20}$

Solution:

$$\begin{aligned} \text{a. } \mathcal{L}^{-1} \frac{2s-5}{9s^2-25} &= \mathcal{L}^{-1} \left[ \frac{2s}{9s^2-25} - \frac{5}{9s^2-25} \right] \\ &= \mathcal{L}^{-1} \left[ \frac{2s}{9(s^2 - (\frac{5}{3})^2)} - \frac{5}{9(s^2 - (\frac{5}{3})^2)} \right] \\ &= \frac{2}{9} \cosh \frac{5}{3} t - \frac{1}{3} \left[ \frac{5/3}{s^2 - (\frac{5}{3})^2} \right] \\ &= \frac{2}{9} \cos \frac{5}{3} t - \frac{1}{3} \sin \frac{5t}{3} \dots \text{Ans.} \end{aligned}$$

$$\begin{aligned} \text{b. } \mathcal{L}^{-1} \frac{s-2}{6s^2+20} &= \mathcal{L}^{-1} \frac{s}{6s^2+20} - \mathcal{L}^{-1} \frac{2}{6s^2+20} \\ &= \frac{1}{6} \mathcal{L}^{-1} \frac{s}{s^2 + \frac{10}{3}} - \frac{1}{3} \mathcal{L}^{-1} \frac{1}{s^2 + \frac{10}{3}} \\ &= \frac{1}{6} \cos \sqrt{\frac{10}{3}} t - \frac{1}{3} \cdot \sqrt{\frac{3}{10}} \mathcal{L}^{-1} \frac{\sqrt{\frac{10}{3}}}{s^2 + \frac{10}{3}} \\ &= \frac{1}{6} \cos \sqrt{\frac{10}{3}} t - \frac{1}{\sqrt{30}} \sin \sqrt{\frac{10}{3}} t \dots \text{Ans.} \end{aligned}$$

B. Multiplication by s

$$\mathcal{L}^{-1} [s F(s)] = \frac{d}{dt} f(t) + f(0) \delta(t)$$

Example 11. Find the inverse laplace trans. of

a.  $\frac{s}{s^2+1}$ , b.  $\frac{s}{4s^2-25}$ , c.  $\frac{3s}{2s+9}$

Solution

$$\begin{aligned} \text{a. } \mathcal{L}^{-1} \frac{1}{s^2+1} &= \sin t \\ \mathcal{L}^{-1} \frac{s}{s^2+1} &= \frac{d}{dt} (\sin t) + \sin(0) \delta(t) \\ &= \cos t \dots \text{Ans.} \end{aligned}$$

$$b. \mathcal{L}^{-1} \frac{1}{4s^2 - 25} = \frac{1}{4} \mathcal{L}^{-1} \frac{1}{s^2 - \frac{25}{4}} = \frac{1}{4} \cdot \frac{2}{5} \mathcal{L}^{-1} \frac{\frac{5}{2}}{s^2 - (\frac{5}{2})^2}$$

$$= \frac{1}{10} \sinh \frac{5}{2} t$$

$$\mathcal{L}^{-1} \frac{5}{4s^2 - 25} = \frac{1}{10} \frac{d}{dt} \sinh \frac{5}{2} t + \frac{1}{10} \sin \frac{5}{2} (0)$$

$$= \frac{1}{10} \left( \frac{5}{10} \right) \cos \frac{5}{2} t = \frac{1}{4} \cosh \frac{5}{2} t \dots \text{Ans.}$$

$$c. \mathcal{L}^{-1} \frac{3s}{2s+9} = \frac{3}{2} \mathcal{L}^{-1} \frac{1}{s + \frac{9}{2}} = \frac{3}{2} e^{-9/2 t}$$

$$= \mathcal{L}^{-1} \frac{3s}{2s+9} = \frac{3}{2} \frac{d}{dt} (e^{-9/2 t}) + \frac{3}{2} e^{-9/2 (0)}$$

$$= \frac{3}{2} \left( -\frac{9}{2} \right) e^{-\frac{11}{2} t} + \frac{3}{2}$$

$$= -\frac{27}{4} e^{-\frac{11}{2} t} + \frac{3}{2} \dots \text{Ans.}$$

C. Division by s (multiplication by  $\frac{1}{s}$ )

$$\mathcal{L}^{-1} \left[ \frac{F(s)}{s} \right] = \int_0^t \mathcal{L}^{-1} [F(s)] dt = \int_0^t f(t) dt$$

Example 12. Find the inverse Laplace transform of.

$$a. \frac{1}{s(s+a)}, \quad b. \frac{1}{s(s^2+1)}$$

Solution:

$$a. \mathcal{L}^{-1} \left( \frac{1}{s+a} \right) = e^{-at}$$

$$\mathcal{L}^{-1} \left[ \frac{1}{s(s+a)} \right] = \int_0^t \mathcal{L}^{-1} \left[ \frac{1}{s+a} \right] dt = \int_0^t e^{-at} dt$$

$$= \left[ \frac{e^{-at}}{-a} \right]_0^t = \frac{e^{-at}}{-a} + \frac{1}{a} = \frac{1}{a} [1 - e^{-at}]$$

Ans.

$$b. \mathcal{L}^{-1} \frac{1}{s^2+1} = \sin t$$

$$\mathcal{L}^{-1} \frac{1}{s} \left[ \frac{1}{s^2+1} \right] = \int_0^t \mathcal{L}^{-1} \left[ \frac{1}{s^2+1} \right] dt$$

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Example 13.

Find the inverse Laplace Trans. of  $\mathcal{L}^{-1} \frac{1}{s(s^2+4)}$   
using  $\int_0^t f(u) du = \frac{F(s)}{s}$ .

Solution:

$$\mathcal{L}^{-1} \frac{1}{s(s^2+4)} = \mathcal{L}^{-1} \frac{F(s)}{s} \Rightarrow \frac{1}{s^2+4}$$

$$\mathcal{L}^{-1} F(s) = f(u) \Rightarrow \mathcal{L}^{-1} \frac{1}{s^2+4} = \frac{1}{2} \mathcal{L}^{-1} \frac{2}{s^2+4} = \frac{1}{2} \sin 2u = f(u)$$

$$\frac{F(s)}{s} = \int_0^t f(u) du = \int_0^t \frac{1}{2} \sin 2u du = -\frac{1}{4} \int_0^t 2 \sin 2u du$$

$$= -\frac{1}{4} \cos 2u \Big|_0^t = \frac{1}{4} (1 - \cos 2t)$$

$$\therefore \mathcal{L}^{-1} \frac{1}{s(s^2+4)} = \frac{1}{4} (1 - \cos 2t) \dots \text{Ans.}$$

Example 14. Evaluate  $\mathcal{L}^{-1} \left[ \frac{1}{s^3(s^2+1)} \right]$

Solution.

$$\text{Since } \mathcal{L}^{-1} \left[ \frac{1}{s^2+1} \right] = \sin t$$

$$\text{we have } \mathcal{L}^{-1} \frac{F(s)}{s} = \int_0^t f(u) du$$

$$\therefore \mathcal{L}^{-1} \frac{1}{s(s^2+1)} = \int_0^t \sin u du = 1 - \cos t \quad \text{and}$$

$$\mathcal{L}^{-1} \frac{1}{s^2(s^2+1)} = \int_0^t (1 - \cos u) du = t - \sin t \quad \text{and}$$

$$\frac{1}{s^3(s^2+1)} = \int_0^t (u - \sin u) du = \frac{t^2}{2} + \cos t - 1$$

$$f(t) = \frac{t^2}{2} + \cos t - 1 \dots \text{Ans.}$$

Note: to check the solution take

$$\mathcal{L}[f(t)] = F(s) \text{ means}$$

$$\mathcal{L}^{-1} \left[ \frac{t^2}{2} + \cos t - 1 \right] = \frac{1}{2} \cdot \frac{2!}{s^3} + \frac{s}{s^2+1} - \frac{1}{s}$$

$$= \frac{1}{s^3} + \frac{s}{s^2+1} - \frac{1}{s}$$

$$F(s) = \frac{s^2+1 + s^4 - s^2(s^2+1)}{s^3(s^2+1)} = \frac{s^2+1 + s^4 - s^4 - s^2}{s^3(s^2+1)}$$

$$= \frac{1}{s^3(s^2+1)} \quad \text{Ans.}$$

Example 15.

$$\mathcal{L}^{-1} \left[ \frac{6}{2s-3} - \frac{3+4s}{9s^2-16} + \frac{8-6s}{16s^2+9} \right]$$

$$= \mathcal{L}^{-1} \left[ \frac{3}{s-\frac{3}{2}} - \frac{3}{9(s^2-\frac{16}{9})} - \frac{4}{9} \cdot \frac{s}{(s^2-\frac{16}{9})} + \frac{8}{16(s^2+\frac{9}{16})} - \frac{6}{16} \cdot \frac{s}{(s^2+\frac{9}{16})} \right]$$

$$= \mathcal{L}^{-1} \left[ \frac{3}{s-\frac{3}{2}} - \frac{1}{3} \mathcal{L}^{-1} \left[ \frac{1}{s^2-\frac{16}{9}} \right] - \frac{4}{9} \mathcal{L}^{-1} \left[ \frac{s}{s^2-\frac{16}{9}} \right] + \frac{1}{2} \mathcal{L}^{-1} \left[ \frac{1}{s^2+\frac{9}{16}} \right] - \frac{3}{8} \mathcal{L}^{-1} \left[ \frac{s}{s^2+\frac{9}{16}} \right] \right]$$

$$= 3e^{\frac{3}{2}t} - \frac{1}{4} \sinh \frac{4}{3}t - \frac{4}{9} \cosh \frac{4}{3}t + \frac{2}{3} \sin \frac{3}{4}t - \frac{3}{8} \cos \frac{3}{4}t \quad \text{Ans.}$$

## D. Partial Fraction Method:

Example 16. Find the inverse transforms of  $\frac{1}{s^2-5s+6}$

Solution; Let us convert the given function into partial fractions.

$$\mathcal{L}^{-1} \left[ \frac{1}{s^2-5s+6} \right] = \mathcal{L}^{-1} \left[ \frac{1}{(s-3)(s-2)} \right]$$

$$= \mathcal{L}^{-1} \left[ \frac{1}{s-3} - \frac{1}{s-2} \right]$$

$$= \mathcal{L}^{-1} \left[ \frac{1}{s-3} \right] - \mathcal{L}^{-1} \left[ \frac{1}{s-2} \right]$$

$$= e^{3t} - e^{2t} \quad \text{Ans.}$$

Example 17. Find the inverse Laplace trans. of

$$\frac{s-1}{s^2-6s+25}$$

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Solution:  $\mathcal{L}^{-1} \left[ \frac{s-1}{s^2-6s+25} \right] = \mathcal{L}^{-1} \left[ \frac{s-1}{s^2-6s+9+16} \right]$   
 $= \mathcal{L}^{-1} \left[ \frac{s-3+2}{(s-3)^2+4^2} \right]$   
 $= \mathcal{L}^{-1} \left[ \frac{s-1}{(s-3)^2+4^2} \right] = \frac{1}{2} \mathcal{L}^{-1} \left[ \frac{4}{(s-3)^2+4^2} \right]$   
 $= e^{3t} \cos 4t + \frac{1}{2} e^{3t} \sin 4t$   
 $\dots \text{Ans.}$

Example 13. Find the inverse Laplace tran. using partial fraction method.  $\mathcal{L}^{-1} \left[ \frac{3s+7}{s^2-2s-3} \right]$

Solution.

$$\mathcal{L}^{-1} \left[ \frac{3s+7}{s^2-2s-3} \right] = \mathcal{L}^{-1} \left[ \frac{3s+7}{(s-3)(s+1)} \right] = \mathcal{L}^{-1} \left[ \frac{A}{s-3} + \frac{B}{s+1} \right]$$

$$= \mathcal{L}^{-1} \left[ \frac{A(s+1)+B(s-3)}{(s-3)(s+1)} \right] = \mathcal{L}^{-1} \left[ \frac{As+A+Bs-3B}{(s-3)(s+1)} \right]$$

$$\Rightarrow 3s+7 = s(A+B) + (A-3B)$$

$$\therefore A+B = 3$$

$$-A+3B = -7$$

$$4B = -4 \quad \text{or} \quad B = -1, \quad A = 4$$

$$\therefore \mathcal{L}^{-1} \left[ \frac{4}{s-3} - \frac{1}{s+1} \right] = 4e^{3t} - e^{-t} \dots \text{Ans.}$$

or : We have the roots,  $s_1 = 3$  &  $s_2 = -1$

$$\therefore A = \left. \frac{3s+7}{(s-3)(s+1)} (s-3) \right|_{s=3} = \frac{16}{4} = 4 = A$$

and

$$B = \left. \frac{3s+7}{(s-3)(s+1)} (s+1) \right|_{s=-1} = \frac{4}{-4} = -1 = B$$

Example 14. Find the inverse Laplace Trans. of

$$\mathcal{L}^{-1} \left[ \frac{2s^2 - 16}{s^3 + 16s} \right] \text{ using partial fraction method.}$$

Solution:

$$\mathcal{L}^{-1} \left[ \frac{2s^2 - 16}{s^3 + 16s} \right] = \mathcal{L}^{-1} \left[ \frac{2s^2 - 16}{s(s-4)(s+4)} \right] = \mathcal{L}^{-1} \left[ \frac{A}{s} + \frac{B}{s-4} + \frac{C}{s+4} \right]$$

$$\mathcal{L}^{-1} \left[ \frac{A(s^2 - 16) + Bs(s+4) + Cs(s-4)}{s(s-4)(s+4)} \right] = \mathcal{L}^{-1} \left[ \frac{As^2 - 16A + Bs^2 + 4Bs + Cs^2 - 4Cs}{s(s-4)(s+4)} \right]$$

$$= \mathcal{L}^{-1} \left[ \frac{s^2(A+B+C) + s(4B-4C) - 16A}{s(s-4)(s+4)} \right] \quad \Downarrow$$

$$2s^2 - 16 = s^2(A+B+C) + s(4B-4C) - 16A$$

$$\therefore A+B+C = 2 \quad \dots (1)$$

$$\& \quad 4B - 4C = 0 \quad \text{or} \quad B = C \quad \dots (2)$$

$$\& \quad -16A = -16 \quad \text{or} \quad A = 1 \quad \dots (3)$$

Sub. 2 & 3 into yield the following

$$A = 1$$

$$B \& C = \frac{1}{2}$$

$$\therefore \mathcal{L}^{-1} \left[ \frac{1}{s} + \frac{1/2}{s-4} + \frac{1/2}{s+4} \right] = 1 + \frac{1}{2} e^{4t} + \frac{1}{2} e^{-4t}$$

$$= 1 + \cosh 4t$$

--- Ans.

Example 15.

$$\mathcal{L}^{-1} \left[ \frac{1}{s^2(s^2+4)} \right] = \mathcal{L}^{-1} \left[ \frac{A}{s^2} + \frac{B}{s} + \frac{Cs+D}{s^2+4} \right]$$

$$\mathcal{L}^{-1} \left[ \frac{A(s^2+4) + Bs(s^2+4) + s^2(Cs+D)}{s^2(s^2+4)} \right] =$$

$$\mathcal{L}^{-1} \left[ \frac{As^2 + 4A + Bs^3 + 4Bs + s^3C + s^2D}{s^2(s^2+4)} \right] =$$

(27)

(9) 15

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$$\mathcal{L}^{-1} \frac{s^3(B+C) + s^2(A+D) + 4Bs + 4A}{s^2(s^2+4)}$$



$$1 = s^3(B+C) + s^2(A+D) + 4Bs + 4A$$

$$\therefore B+C = 0$$

$$A+D = 0$$

$$4B = 0 \Rightarrow B = 0 \Rightarrow C = 0$$

$$4A = 1 \Rightarrow A = \frac{1}{4} \Rightarrow D = -\frac{1}{4}$$

$$\therefore \mathcal{L}^{-1} \left[ \frac{1/4}{s^2} + \frac{0}{s} + \frac{(0)s - 1/4}{s^2+4} \right] = \mathcal{L}^{-1} \frac{1/4}{s} - \frac{1}{4} \mathcal{L}^{-1} \frac{1}{s^2+4}$$

$$= \frac{1}{4} \mathcal{L}^{-1} \frac{1}{s} - \frac{1}{8} \mathcal{L}^{-1} \frac{2}{s^2+4} = \frac{1}{4} - \frac{1}{8} \sin 2t = f(t)$$

Ans.

Example 16.

$$\mathcal{L}^{-1} \left[ \frac{2s^2+11s+5}{s(s+2)(s+3)} \right] = \mathcal{L}^{-1} \left[ \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+3} \right]$$

This function contains three non repeated roots, (0, -2, -3) so:

$$A = \frac{2s^2+11s+5}{s(s+2)(s+3)} \cdot s \Big|_{s=0} = \frac{5}{6} \Rightarrow$$

$$B = \frac{2s^2+11s+5}{s(s+2)(s+3)} \cdot (s+2) \Big|_{s=-2} = \frac{9}{2}$$

$$C = \frac{2s^2+11s+5}{s(s+2)(s+3)} \cdot (s+3) \Big|_{s=-3} = -\frac{10}{3}$$

$$\therefore \mathcal{L}^{-1} \left[ \frac{5/6}{s} + \frac{9/2}{s+2} - \frac{10/3}{s+3} \right]$$

$$= \frac{5}{6} + \frac{9}{2} e^{-2t} - \frac{10}{3} e^{-3t} \quad \text{Ans.}$$

Example 17.

$$\begin{aligned}\mathcal{L}^{-1} \frac{1}{s^3(s^2+1)} &= \mathcal{L}^{-1} \left[ \frac{A}{s^3} + \frac{B}{s^2} + \frac{C}{s} + \frac{Ds+E}{s^2+1} \right] \\ &= \mathcal{L}^{-1} \frac{A(s^2+1) + Bs(s^2+1) + Cs^2(s^2+1) + s^3(Ds+E)}{s^3(s^2+1)} \\ &= \mathcal{L}^{-1} \frac{As^2+A+Bs^3+Bs+Cs^4+Cs^2+Ds^4+ES^3}{s^3(s^2+1)}\end{aligned}$$



$$\begin{aligned}\therefore 1 &= s^4(C+D) + s^3(B+E) + s^2(A+C) + Bs + A \\ \left. \begin{aligned} C+D &= 0 \\ B+E &= 0 \\ A+C &= 0 \end{aligned} \right\} & \begin{aligned} B &= 0 \Rightarrow E=0 \\ A &= 1 \Rightarrow C=-1 \Rightarrow D=1 \end{aligned}\end{aligned}$$

$$\therefore \mathcal{L}^{-1} \left[ \frac{1}{s^3} + \frac{0}{s^2} - \frac{1}{s} + \frac{s}{s^2+1} \right] =$$

$$\frac{1}{2} t^2 - 1 + \cos t = f(t) \quad \text{--- Ans.}$$

Example 18. Find

$$\mathcal{L}^{-1} \left[ \frac{2s^2-4}{(s+1)(s-2)(s-3)} \right]$$

we have

$$\mathcal{L}^{-1} \left[ \frac{2s^2-4}{(s+1)(s-2)(s-3)} \right] = \frac{A}{s+1} + \frac{B}{s-2} + \frac{C}{s-3} \quad \dots (1)$$

we have the roots:  $s = -1, s = 2, s = 3$   
and

$$A = \frac{2s^2-4}{(s+1)(s-2)(s-3)} \cdot (s+1) \Big|_{s=-1} = \frac{2 \cdot 1 - 4}{(-1-2)(-1-3)} = \frac{-2}{12}$$

$$A = -\frac{1}{6}$$

$$B = \left. \frac{2s^2 - 4}{(s+1)(s-2)(s-3)} \cdot (s-2) \right|_{s=2} = \frac{2 \times 4 - 4}{(2+1)(2-3)} = \frac{4}{-3}$$

$$B = -\frac{4}{3}$$

$$C = \left. \frac{2s^2 - 4}{(s+1)(s-2)(s-3)} \cdot (s-3) \right|_{s=3} = \frac{2 \times 9 - 4}{(3+1)(3-2)} = \frac{14}{4}$$

$$C = \frac{7}{2}$$

or we can find A, B & C as follow:

Multiply both sides of (1) by  $(s+1)$  and let  $s \rightarrow -1$ ; then

$$A = \lim_{s \rightarrow -1} \frac{2s^2 - 4}{(s-2)(s-3)} = -\frac{1}{6}$$

Multiply both sides of (1) by  $s-2$  and let  $s \rightarrow 2$ ; then

$$B = \lim_{s \rightarrow 2} \frac{2s^2 - 4}{(s+1)(s-3)} = -\frac{4}{3}$$

Multiply both sides of (1) by  $s-3$  and let  $s \rightarrow 3$ ; then

$$C = \lim_{s \rightarrow 3} \frac{2s^2 - 4}{(s+1)(s-2)} = \frac{7}{2}$$

Thus

$$\begin{aligned} \mathcal{L}^{-1} \left[ \frac{2s^2 - 4}{(s+1)(s-2)(s-3)} \right] &= \mathcal{L}^{-1} \left[ \frac{-1/6}{s+1} + \frac{-4/3}{s-2} + \frac{7/2}{s-3} \right] \\ &= -\frac{1}{6} e^{-t} - \frac{4}{3} e^{2t} + \frac{7}{2} e^{3t} \dots \text{Ans} \end{aligned}$$

Example 19. Find

$$\mathcal{L}^{-1} \left[ \frac{5s^2 - 15s - 11}{(s+1)(s-2)^3} \right]$$

Solution: First partial fraction.

$$\frac{5s^2 - 15s - 11}{(s+1)(s-2)^3} = \frac{A}{s+1} + \frac{B}{(s-2)^3} + \frac{C}{(s-2)^2} + \frac{D}{(s-2)} \quad (1)$$

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A procedure analogous to that of problem 18 can be used to find A & B.

Multiply both sides of (1) by  $s+1$  and let  $s \rightarrow -1$ ; then

$$A = \lim_{s \rightarrow -1} \frac{5s^2 - 15s - 11}{(s-2)^3} = -\frac{1}{8}$$

Multiply both sides of (1) by  $(s-2)^3$  and let  $s \rightarrow 2$ ; then

$$B = \lim_{s \rightarrow 2} \frac{5s^2 - 15s - 11}{s+1} = -7$$

The method fail to determine C & D, however, since we know A and B, we have from (1),

$$\frac{5s^2 - 15s - 11}{(s+1)(s-2)^3} = \frac{-1/8}{s+1} + \frac{-7}{(s-2)^3} + \frac{C}{(s-2)^2} + \frac{D}{s-2} \quad (2)$$

To determine C and D we can substitute two values for s, say  $s=0$  and  $s=1$ , from which we find respectively,

$$\frac{11}{8} = -\frac{1}{8} + \frac{7}{8} + \frac{C}{4} - \frac{D}{2}, \quad \frac{21}{2} = -\frac{1}{6} + 7 + C - D$$

i.e.,  $3C - 6D = 10$  and  $3C - 3D = 11$ , from which,

$$C = 4, \quad D = 1/3. \text{ Thus}$$

$$\mathcal{L}^{-1} \left[ \frac{5s^2 - 15s - 11}{(s+1)(s-2)^3} \right] = \mathcal{L}^{-1} \left[ \frac{-1/8}{s+1} + \frac{-7}{(s-2)^3} + \frac{4}{(s-2)^2} + \frac{1/3}{s-2} \right]$$

$$= -\frac{1}{8} e^{-t} - \frac{7}{2} t^2 e^{2t} + 4t e^{2t} + \frac{1}{3} e^{2t}$$

Ans

Example 20. Find

$$\mathcal{L}^{-1} \frac{s^2 + 2s + 3}{(s^2 + 2s + 2)(s^2 + 2s + 5)}$$

solution;

$$\mathcal{L}^{-1} \frac{s^2 + 2s + 3}{(s^2 + 2s + 2)(s^2 + 2s + 5)} = \mathcal{L}^{-1} \left[ \frac{As + B}{s^2 + 2s + 2} + \frac{Cs + D}{s^2 + 2s + 5} \right]$$

$$= \mathcal{L}^{-1} \frac{(As + B)(s^2 + 2s + 5) + (Cs + D)(s^2 + 2s + 2)}{(s^2 + 2s + 5)(s^2 + 2s + 2)}$$

$$= \mathcal{L}^{-1} \frac{As^3 + 2As^2 + 5As + Bs^2 + 2Bs + 5B + Cs + 2Cs^2 + 2Cs + Ds^2 + 2Ds + 2D}{(s^2 + 2s + 2)(s^2 + 2s + 5)}$$

$$\therefore s^2 + 2s + 3 = s^3(A+C) + s^2(2A+B+2C+D) + s(5A+2B+2C+2D) + (5B+2D)$$

$$A + C = 0 \quad \dots (1) \Rightarrow A = -C$$

$$2A + B + 2C + D = 1 \quad \dots (2)$$

$$5A + 2B + 2C + 2D = 2 \quad \dots (3)$$

From equation 1, 2, 3

$$5B + 2D = 3 \Rightarrow B = \frac{3-2D}{5} \quad \dots (4)$$

from 2, 3 and 4

$$D = \frac{2}{3} \quad \& \quad B = \frac{1}{3}$$

sub. the values of B & D into equations 2 & 3

$$2A + \frac{1}{3} + 2C + \frac{2}{3} = 1 \Rightarrow 2A + 2C = 0$$

$$5A + \frac{2}{3} + 2C + \frac{4}{3} = 2 \Rightarrow 5A + 2C = 0$$

$$C = A = 0$$

$$\therefore \mathcal{L} \left[ \frac{1/3}{s^2 + 2s + 2} + \frac{2/3}{s^2 + 2s + 5} \right] = \mathcal{L}^{-1} \frac{1/3}{(s^2 + 2s + 1) - 1 + 2} + \mathcal{L}^{-1} \frac{2/3}{(s^2 + 2s + 1) - 1 + 4}$$

$$= \frac{1}{3} \mathcal{L}^{-1} \frac{1}{(s+1)^2 + 1} + \frac{2}{3} \mathcal{L}^{-1} \frac{1}{(s+1)^2 + 4}$$

$$= \frac{1}{3} e^{-t} \sin t + \frac{2}{3} \times \frac{1}{2} e^{-t} \sin 2t$$

$$= \frac{1}{3} e^{-t} (\sin t + \sin 2t) \quad \dots \dots \text{Ans.}$$

## problems

~~Find the inverse Laplace Transform of the following:~~ Find the inverse Laplace Transform of the following;

$$1 - \frac{s^2 + 6s + 6}{(s+1)(s^2 + 4s + 3)}$$

$$2 - \frac{12s - 28}{(s-5)(s+3)(s-1)}$$

$$3 - \frac{3s^2 + 8s + 9}{(s+3)(s+1)^2}$$

$$4 - \frac{s^3 + 4s^2 + 4s - 5}{(s+3)(s+1)^3}$$

$$5 - \frac{2s^2}{(s^2+1)(s-1)^2}$$

$$6 - \frac{5}{s^2 + 6s + 1}$$

$$7 - \frac{s+1}{s^2 + s + 1}$$

$$8 - \frac{12}{4 - 3s}$$

$$9 - \frac{1}{2s - 5}$$

$$10 - \frac{3(s^2 - 1)^2}{2s^5} + \frac{4s - 18}{9 - s^2}$$

$$11 - \frac{s}{(s+1)^5}$$

$$12 - \frac{3s - 14}{s^2 - 4s + 8}$$

$$13 - \frac{8s + 20}{s^2 - 12s + 32}$$

$$\mathcal{L}^{-1} \left\{ \frac{(s+1)-1}{(s+1)^5} \right\} = \mathcal{L}^{-1} \left\{ \frac{(s+1)}{(s+1)^5} \right\} - \mathcal{L}^{-1} \left\{ \frac{1}{(s+1)^5} \right\}$$

Notes:

Pages 16, 17, 18, 19 and 20 describes the methods of partial fraction decomposition.

## Partial Fractions Decomposition

**Case 1:**  $q(x)$  is a product of distinct linear factors.

Let us assume that  $q(x)$  is a product of  $n$  distinct linear factors that is

$$q(x) = (a_1x + b_1)(a_2x + b_2) \dots (a_nx + b_n)$$

Then,

$$\frac{p(x)}{q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \dots + \frac{A_n}{a_nx + b_n}$$

Finding the decomposition amounts to finding the coefficients  $A_1, \dots, A_n$ . This can be done two different ways. We illustrate this in the examples below.

**Remark 1**  $p(x)$  does not play a role in the way the decomposition is written.

**Example 2** Find the decomposition for  $\frac{x}{x^2 + 2x - 3}$

We begin by factoring the denominator. We obtain  $\frac{x}{(x+3)(x-1)}$ . According to the decomposition we wrote above, we have:

$$\frac{x}{(x+3)(x-1)} = \frac{A}{x+3} + \frac{B}{x-1}$$

We need to find  $A$  and  $B$ . Multiplying each side by the denominator of the fraction on the left and simplifying, we obtain:

$$\begin{aligned} x &= A(x-1) + B(x+3) \\ &= Ax - A + Bx + 3B \\ &= (A+B)x - A + 3B \end{aligned}$$

Two polynomials are equal if their corresponding coefficients are equal. This gives us the following system:

$$\begin{cases} A + B = 1 \\ -A + 3B = 0 \end{cases}$$

The solution of this system is:  $\begin{cases} A = \frac{3}{4} \\ B = \frac{1}{4} \end{cases}$  Thus, we have

$$\frac{x}{x^2 + 2x - 3} = \frac{1}{4} \left( \frac{3}{x+3} + \frac{1}{x-1} \right)$$

There is an easier way which works in this case. We show it on the next example.

**Example 3** Find the decomposition for  $\frac{x^2 - x}{(x-1)(x+5)(x-3)}$

The denominator is already factored. The decomposition is:

$$\frac{x^2 - x}{(x-1)(x+5)(x-3)} = \frac{A}{x-1} + \frac{B}{x+5} + \frac{C}{x-3}$$

We need to find  $A$ ,  $B$ , and  $C$ . We begin the same way, we multiply each side by the denominator of the fraction on the left, and simplify. We obtain:

$$x^2 - x = A(x+5)(x-3) + B(x-1)(x-3) + C(x-1)(x+5)$$

Then we notice that since the above equality is true for every  $x$ , it will be true for specific values of  $x$ . We select values for  $x$  which will make all but one of the coefficients go away. We will then be able to solve for that coefficient. More precisely,

- When  $x = 1$ , we obtain:

$$0 = A(6)(-2)$$

$$A = 0$$

- When  $x = -5$ , we obtain

$$30 = B(-6)(-8)$$

$$B = \frac{30}{48}$$

$$B = \frac{5}{8}$$

- When  $x = 3$ , we obtain

$$6 = C(2)(8)$$

$$C = \frac{3}{8}$$

Therefore, the decomposition is:

$$\frac{x^2 - x}{(x-1)(x+5)(x-3)} = \frac{1}{8} \left( \frac{5}{x+5} + \frac{3}{x-3} \right)$$

**Case 2:  $q(x)$  is a product of linear factors, some being repeated**

The factors which are not repeated will be decomposed as above. Suppose that  $q(x)$  also contains  $(ax+b)^m$  that is  $ax+b$  is repeated  $m$  times. The decomposition for this factor will be

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \dots + \frac{A_m}{(ax+b)^m}$$

**Example 4** Find a decomposition for  $\frac{x^2 - 2}{(x-2)(x+1)^3}$

The decomposition is:

$$\frac{x^2 - 2}{(x-2)(x+1)^3} = \frac{A}{x-2} + \frac{B}{x+1} + \frac{C}{(x+1)^2} + \frac{D}{(x+1)^3}$$

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Then, we proceed as before. We multiply each side by the denominator of the fraction on the left and simplify. We obtain:

$$x^2 - 2 = A(x+1)^3 + B(x-2)(x+1)^2 + C(x-2)(x+1) + D(x-2)$$

We need to find A, B, C, and D. We can use either of the methods described in the first case.

- If  $x = 2$ , we get

$$2 = 27A$$

$$A = \frac{2}{27}$$

- If  $x = -1$ , we get

$$-1 = -3D$$

$$D = \frac{1}{3}$$

We still have to find B and C. For this, we choose two more values for x and write the corresponding system. We now know A and D, so we can use the value we found for them.

- If  $x = 1$ , we get

$$-1 = 8A - 4B - 2C - D$$

$$= \frac{16}{27} - 4B - 2C - \frac{1}{3}$$

$$4B + 2C = \frac{16}{27} - \frac{9}{27} + \frac{27}{27}$$

$$4B + 2C = \frac{34}{27}$$

- If  $x = 0$ , we get

$$-2 = A - 2B - 2C - 2D$$

$$= \frac{2}{27} - 2B - 2C - \frac{2}{3}$$

$$2B + 2C = \frac{2}{27} - \frac{18}{27} + \frac{54}{27}$$

$$2B + 2C = \frac{38}{27}$$

- Thus, we need to solve

$$\begin{cases} 4B + 2C = \frac{34}{27} \\ 2B + 2C = \frac{38}{27} \end{cases}$$

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The solution is :  $\{B = -\frac{2}{27}, C = \frac{7}{9}\}$ . Putting all this together, we get

$$\frac{x^2 - 2}{(x - 2)(x + 1)^3} = \frac{2}{27(x - 2)} - \frac{2}{27(x + 1)} + \frac{7}{9(x + 1)^2} + \frac{1}{3(x + 1)^3}$$

**Case 3:**  $q(x)$  is a product of distinct irreducible quadratic factors.

Recall that a term is called irreducible if it cannot be factored any further. Thus  $x^2 + x + 1$  is irreducible, so is  $x^2 + 1$ . Be careful,  $(x + 1)^2$  is not considered a quadratic term. You must think of it as a linear term appearing twice. We can generalize what we did in the previous two cases as follows. Instead of thinking of linear factors, think that when we write the decomposition, the degree of the term in the numerator is 1 less than the degree of the term in the denominator. When we had linear factors in the denominator, it meant that we had to have terms of degree 0 in the numerator, that is we had constant terms. If the denominator contains irreducible quadratic factors, then the numerator will contain linear terms. We look at an example to see how the decomposition is written.

**Example 5** Decompose  $\frac{x}{(x^2 + 1)(x^2 + 2)}$

If we apply a method similar to that of case 1, the decomposition will contain two fractions, one for each irreducible factor. However, since the degree of the denominator is now 2, the degree of the numerator will be 1, that is we will have linear terms. Recall that a linear term is of the form  $ax + b$ . Thus,

$$\frac{x}{(x^2 + 1)(x^2 + 2)} = \frac{Ax + B}{x^2 + 1} + \frac{Cx + D}{x^2 + 2}$$

To solve, we proceed as above. First, we multiply each side by the denominator of the fraction on the left, and simplify. We obtain:

$$x = (Ax + B)(x^2 + 2) + (Cx + D)(x^2 + 1)$$

We then pick 4 different values for  $x$  to get a system of 4 equations, which we solve. The answer is:

$$\frac{x}{(x^2 + 1)(x^2 + 2)} = \frac{x}{x^2 + 1} - \frac{x}{x^2 + 2}$$

**Case 4:**  $q(x)$  is a product of irreducible quadratic factors, some being repeated

This is similar to case 2, with linear terms in the numerator and quadratic terms in the denominator.

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Example 6 Write the decomposition for  $\frac{2x-1}{(x^2+x+1)^3}$

$$\frac{2x-1}{(x^2+x+1)^3} = \frac{Ax+B}{x^2+x+1} + \frac{Cx+D}{(x^2+x+1)^2} + \frac{Ex+F}{(x^2+x+1)^3}$$

We would find the coefficients as above.

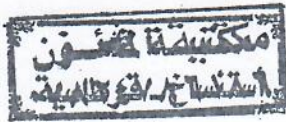
1.6 General Case:  $q(x)$  is a mixture of the above

Example 7 Decompose  $\frac{2x-1}{(x-1)^2(x^2+x+1)^3}$

$$\begin{aligned} \frac{2x-1}{(x-1)^2(x^2+x+1)^3} &= \frac{A}{x-1} + \frac{B}{(x-1)^2} \\ &+ \frac{Cx+D}{x^2+x+1} + \frac{Ex+F}{(x^2+x+1)^2} + \frac{Gx+H}{(x^2+x+1)^3} \end{aligned}$$

The answer is

$$\begin{aligned} \frac{2x-1}{(x-1)^2(x^2+x+1)^3} &= \frac{1}{27(x-1)^2} - \frac{1}{27(x-1)} \\ &+ \frac{1}{27} \frac{x+1}{x^2+x+1} - \frac{1}{9} \frac{1}{(x^2+x+1)^2} - \frac{1}{3} \frac{x+3}{(x^2+x+1)^3} \end{aligned}$$



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# Laplace Transform

The main application of Laplace transformation for will be solving some differential equations, the Laplace Trans. converts integral and differential equations into algebraic equations. [most diff. eq. with initial values have a unique solution].

We define the Laplace Transform of a function  $f(t)$ , as:

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt \quad \text{for } s \in \mathbb{C} \text{ (complex num.)}$$

and we use  $F(s) = \mathcal{L}\{f(t)\}$

Definition: Given a function  $f(t)$ ,  $t \geq 0$ , its Laplace Trans.

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt = \lim_{A \rightarrow \infty} \int_0^A e^{-st} f(t) dt$$

We say the transform converges if the limit exists and diverges if not

\* Find the Laplace Trans. using definition

1-  $f(t) = 1$  for  $t > 0$

Solution:  $F(s) = \mathcal{L}\{f(t)\} = \mathcal{L}\{1\} = \int_0^{\infty} e^{-st} (1) dt = -\frac{1}{s} \left[ e^{-st} \right]_0^{\infty}$   
 $= -\frac{1}{s} \left[ \frac{1}{e^{st}} \right]_0^{\infty} = \frac{1}{s}$

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2-  $f(t) = 3$

Solution:-  $F(s) = \mathcal{L}\{f(t)\} = \mathcal{L}\{3\} = \int_0^{\infty} e^{-st} (3) dt = \frac{3}{s}$

In general If  $\boxed{f(t) = a \Rightarrow F(s) = \frac{a}{s}}$

3-  $f(t) = u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$  unit step function

Solu:-  $\mathcal{L}\{u(t)\} = \int_0^{\infty} e^{-st} (1) dt + \int_0^{\infty} e^{-st} (1) dt$   
 $= \int_0^{\infty} e^{-st} dt \Rightarrow [ex(1)]$

4-  $f(t) = t$

Solu:-  $\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt = \int_0^{\infty} e^{-st} t dt$  using (u)

$\int u dv = uv - \int v du$

$\int_0^{\infty} e^{-st} t dt = \frac{-t}{s} e^{-st} \Big|_0^{\infty} - \int_0^{\infty} \frac{1}{s} e^{-st} dt$

$= -\frac{1}{s^2} \left( \frac{1}{e} \Big|_0^{\infty} \right) = \frac{1}{s^2}$

In general

$\boxed{\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}}$

$n \geq 1$

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$$5. f(t) = t^3$$

(3)

Solu:  $\mathcal{L}(t^3) = \frac{3!}{s^4} = \frac{6}{s^4}$

6- The Laplace trans. of  $(\sin at)$  and  $(\cos at)$

Solu: we can find  $\mathcal{L}(\sin at)$  and  $\mathcal{L}(\cos at)$  using  $(u dv)$  twice

or by the Euler's Formula,  $e^{iat} = \cos at + i \sin at$

$$\rightarrow \mathcal{L}[e^{iat}] = \int_0^{\infty} e^{-st} e^{iat} dt = \int_0^{\infty} e^{-(s-ia)t} dt = \frac{1}{s-ia}$$

$$\therefore \mathcal{L}[e^{iat}] = \frac{1}{s-ia} \cdot \frac{s+ia}{s+ia} = \frac{s+ia}{s^2+a^2} = \frac{s}{s^2+a^2} + i \frac{a}{s^2+a^2}$$

Comparing the real and the imaginary parts we get

$$\boxed{\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}} \quad , \quad \boxed{\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}} \quad s > 0$$

$$\therefore \mathcal{L}\{\cos 5t\} = \frac{s}{s^2+25} \quad \text{and} \quad \mathcal{L}\{\sin 3t\} = \frac{3}{s^2+9}$$

H.w find  $\mathcal{L}\{\cosh at\}$  and  $\mathcal{L}\{\sinh at\}$

## Elementary Properties of Laplace Trans.

Suppose  $a, b \in \mathbb{C}$  and  $f, g$  functions for which Laplace Transform exists then:-

$$1. \mathcal{L}\{af(t) + bg(t)\} = aF(s) + bG(s) \text{ [Linearity]}$$

$$2. \mathcal{L}\{e^{at} f(t)\} = F(s-a) \text{ [First shifting property]}$$

3. Suppose  $f$  and all its derivatives up to and including order " $n$ " are continuous on  $[0, \infty)$  and each derivative has a Laplace transformation, then

$$\mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s)$$

$$4. \text{ If } \mathcal{L}\{f(t)\} = F(s), \text{ then } \mathcal{L}\left\{\int_0^t f(u) du\right\} = \frac{F(s)}{s}$$

$$5. \mathcal{L}\{f(at)\} = \frac{1}{a} F\left(\frac{s}{a}\right) \text{ [Time scaling property]}$$

$$6. \text{ First derivative: } \mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$$

$$\text{Second derivative } \mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$$

and higher order derivatives:-

$$\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \dots - s f^{(n-2)}(0) - f^{(n-1)}(0)$$

# Examples

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1- Find  $F(s)$  if  $f(t) = t e^{2t}$

Solution: - using  $\mathcal{L}\{e^{at} f(t)\} = F(s-a)$

$$\mathcal{L}\{e^{-2t} t\} \Rightarrow f(t) = t \Rightarrow F(s) = \frac{1}{s^2} \Rightarrow F(s+2) = F(s+2) = \frac{1}{(s+2)^2}$$

or using  $\mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s)$

2- Find  $\mathcal{L}\{t \sin(2t)\}$

Solu: using  $\mathcal{L}\{t^n f(t)\}$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{\sin 2t\} = \frac{2}{s^2+4} \Rightarrow (-1)^n F^{(n)}(s) = \frac{4s}{(s^2+4)^2}$$

3- If  $\mathcal{L}[f(t)] = \frac{s}{s^3+2}$  Find  $\mathcal{L}\left[-\frac{t}{e} f(2t)\right]$

Solu: using  $\mathcal{L}\{f(at)\} = \frac{1}{a} F\left(\frac{s}{a}\right)$

$$\mathcal{L}\{f(2t)\} = \frac{1}{2} F\left(\frac{s}{2}\right) = \frac{1}{2} \frac{\frac{s}{2}}{\left(\frac{s}{2}\right)^3+2} = \frac{1}{2} \frac{\frac{s}{2}}{\frac{s^3}{8}+2} = \frac{2s}{s^3+16}$$

$$\mathcal{L}\left\{-\frac{t}{e} f(2t)\right\} = \frac{2(s+1)}{(s+1)^3+16}$$

4- Find  $\mathcal{L}[t^4 - 4t + 5 + 3 \sin 2t]$

Solution  $\mathcal{L} t^4 - 4 \mathcal{L} t + \mathcal{L} 5 + 3 \mathcal{L} \sin 2t = \frac{24}{s^5} - \frac{4}{s^2} + \frac{5}{s} + \frac{6}{s^2+4}$

5- Find  $\mathcal{L}\{4t \cos(7t)\}$

Solu:  $4 \mathcal{L}\{t \cos(7t)\} = (-1) \frac{d}{ds} \frac{s}{s^2+49} = (-1) \left[ \frac{s^2+49+2s^2}{(s^2+49)^2} \right]$

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$$= \frac{s^2 - 49}{(s^2 + 49)^2} \Rightarrow 4 \mathcal{L}\{f(t)\} \Rightarrow \frac{4(s^2 - 49)}{(s^2 + 49)^2}$$

6 - Find the Laplace Trans.  $f(t) = \cos^2 t$

Solus  $\cos^2 t = \frac{1 + \cos 2t}{2} \Rightarrow f(t) = \frac{1}{2}(1 + \cos 2t)$

Taking Laplace  $\mathcal{L}\{f(t)\} = \frac{1}{2} \mathcal{L}\{1 + \cos 2t\} = \frac{1}{2} \left[ \frac{1}{s} + \frac{s}{s^2 + 4} \right]$   
 $= \frac{s^2 + 2}{s(s^2 + 4)}$

or we can solve using derivative

$$\mathcal{L}\{f'(t)\} = s \mathcal{L}\{f(t)\} - f(0)$$

$$f'(t) = -2 \cos t \sin t = -\sin 2t = \cancel{\mathcal{L}\{f(t)\}}$$

$$\therefore \mathcal{L}\{f'(t)\} = \mathcal{L}\{-\sin 2t\} = -\frac{2}{s^2 + 4}, \quad f(0) = \cos^2(0) = 1$$

$$\Rightarrow -\frac{2}{s^2 + 4} = s \mathcal{L}\{f(t)\} - 1 \Rightarrow s \mathcal{L}\{f(t)\} = 1 - \frac{2}{s^2 + 4}$$

$$s \mathcal{L}\{f(t)\} = \frac{s^2 + 2}{s^2 + 4} \Rightarrow \mathcal{L}\{f(t)\} = \frac{s^2 + 2}{s(s^2 + 4)}$$

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7- Find  $\mathcal{L}\{e^{-t} \cos 2t\}$

Soln:  $\mathcal{L}\{e^{-t} \cos 2t\} = \mathcal{L}\{e^{at} f(t)\} = F(s-a)$

$\therefore f(t) = \cos 2t \Rightarrow F(s) = \frac{s}{s^2+4}$

$\therefore F(s+1) = \frac{s+1}{(s+1)^2+4} = \frac{s+1}{s^2+2s+5}$

8-  $\mathcal{L}\left\{\int_0^t \sin 2u du\right\}$

Soln: Since  $\mathcal{L}\int_0^t f(u) du = \frac{F(s)}{s}$

$\therefore f(t) = \sin 2u \Rightarrow F(s) = \frac{2}{s^2+4} \Rightarrow \frac{F(s)}{s} = \frac{2}{s(s^2+4)}$

9-  $\mathcal{L}\{t^2 e^{2t}\}$

Soln:  $\mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s)$

$f(t) = e^{2t} \Rightarrow F(s) = \frac{1}{s-2}$

$\therefore (-1)^2 F''(s) = \frac{2}{(s-2)^3}$

(10)  $\mathcal{L}\{e^{4t} \cosh 5t\}$  (11)  $\mathcal{L}\{e^{-2t} (3 \cos 6t - 5 \sin 6t)\}$

H.W

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12-  $f(t) = [4t \cos 7t]$  Find  $F(s)$

Solu:  $\mathcal{L}\{f(t)\} = \mathcal{L}\{4t \cos 7t\} = 4 \mathcal{L}\{t \cos 7t\}$

$$\mathcal{L}\{\cos 7t\} = \frac{s}{s^2 + 49} = F(s) \Rightarrow \frac{d}{ds} F(s) = \frac{s^2 + 49 - 2s^2}{(s^2 + 49)^2}$$

$$\therefore (-4) F'(s) = \frac{4(s^2 - 49)}{(s^2 + 49)^2}$$

13- Find the Laplace transform of the following functions.

(a)  $f(t) = \int_0^t \cos(4t) dt$

Solu:  $f(t) = \cos 4t \Rightarrow F(s) = \frac{s}{s^2 + 16} \Rightarrow \frac{F(s)}{s} = \frac{1}{s^2 + 16}$

(b)  $f(t) = \int_0^t e^{2t} \cos(3t) dt$

Solu:

$$f(t) = e^{2t} \cos 3t \Rightarrow F(s) = \frac{s-2}{(s-2)^2 + 9}$$

$$\text{and } \frac{F(s)}{s} = \frac{s-2}{s[(s-2)^2 + 9]}$$

(c)  $f(t) = \int_0^t t e^{-3t} dt$

Solu:  $f(t) = t e^{-3t} = e^{-3t} t \Rightarrow F(s) = \frac{1}{(s+3)^2}$

$$\text{and } \frac{F(s)}{s} = \frac{1}{s(s+3)^2}$$

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$$\textcircled{d} \quad f(t) = \int_0^t \sin(at) \cos(at) dt$$

Solu: we have  $\sin 2t = 2 \sin t \cos t$

$$\text{or } \sin 2at = 2 \sin at \cos at$$

$$\Rightarrow \frac{\sin 2at}{2} = \sin at \cos at$$

$$f(t) = \frac{\sin 2at}{2} \Rightarrow F(s) = \frac{2a}{2(s^2 + 4a^2)} \quad \text{and } \frac{F(s)}{s} = \frac{a}{s(s^2 + 4a^2)}$$

$$14- f(t) = e^{-4t} \quad \text{Find } F(s)$$

Solu: using  $\mathcal{L}[F(at)] = \frac{1}{a} F\left(\frac{s}{a}\right)$

$$f(at) = f(-4t) = e^{-4t} \Rightarrow F(t) = \frac{t}{e} \Rightarrow F(s) = \frac{1}{s-1}$$

$$\text{and } F\left(\frac{s}{a}\right) = \frac{1}{\frac{s}{-4} - 1} = \frac{1}{\frac{s - -4}{-4}} = \frac{-4}{s+4} \quad \text{and } \frac{1}{a} F\left(\frac{s}{a}\right) = \frac{-4}{4(s+4)}$$

$$\text{OR } f(t) = e^{-4t} \Rightarrow F(s) = \frac{1}{s+4}$$

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## 1 Trigonometric Identities

$$\sin(x \pm \frac{\pi}{2}) = \pm \cos x \quad *$$

$$\cos(x \pm \frac{\pi}{2}) = \mp \sin x \quad *$$

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y \quad *$$

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y \quad *$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$$

$$2 \sin x \sin y = \cos(x - y) - \cos(x + y) \quad *$$

$$2 \cos x \cos y = \cos(x - y) + \cos(x + y) \quad *$$

$$2 \sin x \cos y = \sin(x - y) + \sin(x + y) \quad *$$

$$\sin x + \sin y = 2 \sin \frac{(x + y)}{2} \cos \frac{(x - y)}{2}$$

$$\sin x - \sin y = 2 \cos \frac{(x + y)}{2} \sin \frac{(x - y)}{2}$$

$$\cos x + \cos y = 2 \cos \frac{(x + y)}{2} \cos \frac{(x - y)}{2}$$

$$\cos x - \cos y = -2 \sin \frac{(x + y)}{2} \sin \frac{(x - y)}{2}$$

$$\tan x \pm \tan y = \frac{\sin(x \pm y)}{\cos x \cos y} \quad |$$

$$\sin 2x = 2 \sin x \cos x \quad *$$

$$\cos 2x = \cos^2 x - \sin^2 x \quad *$$

$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$$

$$2 \sin^2 x = 1 - \cos 2x \quad * \quad |$$

$$2 \cos^2 x = 1 + \cos 2x \quad *$$

$$\cos(\phi + 2\pi) = \cos \phi$$

$$\sin(\phi + 2\pi) = \sin \phi$$

$$\tan(x) = \frac{\sin(x)}{\cos(x)}$$

$$\cot(x) = \frac{\cos(x)}{\sin(x)}$$

$$\sec(x) = \frac{1}{\cos(x)}$$

$$\csc(x) = \frac{1}{\sin(x)}$$

$$4 \sin^3 x = 3 \sin x - \sin 3x$$

$$4 \cos^3 x = 3 \cos x + \cos 3x$$

$$8 \sin^4 x = 3 - 4 \cos 2x + \cos 4x$$

$$8 \cos^4 x = 3 + 4 \cos 2x + \cos 4x$$

$$a \cos x - b \sin x = r \cos(x + \theta)$$

$$\text{where } r = \sqrt{a^2 + b^2}$$

$$\theta = \arctan\left(\frac{b}{a}\right)$$

$$a = r \cos \theta$$

$$b = r \sin \theta$$

## 2 Trigonometric Functions

$$\sin x = \frac{e^{jx} - e^{-jx}}{2j} \quad *$$

$$\cos x = \frac{e^{jx} + e^{-jx}}{2} \quad *$$

$$\tan x = \frac{\sin x}{\cos x} \quad *$$

$$\cos^2 x + \sin^2 x = 1 \quad *$$

$$e^{\pm jx} = \cos x \pm j \sin x \quad *$$

## 3 Hyperbolic Functions

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad *$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \quad *$$

$$\tanh x = \frac{\sinh x}{\cosh x} \quad *$$

$$\cosh^2 x - \sinh^2 x = 1 \quad *$$

$$e^{\pm x} = \cosh x \pm \sinh x \quad *$$

$$\operatorname{arcsinh} x = \ln(x + \sqrt{x^2 + 1})$$

$$\operatorname{arccosh} x = \ln(x + \sqrt{x^2 - 1}), \quad x \geq 1$$

$$\operatorname{artanh} x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right), \quad |x| < 1$$

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