

Complex Numbers



Definitions

Complex Number

A **complex number** is a number $z = a + bi$, where a and b are real numbers

a is the real part of the complex number

b is the imaginary part of the complex number

$$i = \sqrt{-1}$$

Let $i^2 = -1$.

$$\therefore i = \sqrt{-1}.$$

Complex numbers are often denoted by z .

Just as $\mathbb{R}$ is the set of real numbers, $\mathbb{C}$ is the set of complex numbers. If z is a complex number, z is of the form

$$z = x + iy \in \mathbb{C}, \text{ for some } x, y \in \mathbb{R}.$$

$$z = x + iy$$

↑

real part

↙

imaginary part.

e.g. $3 + 4i$ is a complex number.

question: Is 9 a real number or a complex number?

Possible answers:

1. real number
2. complex number
3. both
4. neither

Answer: Both, because 9 can be identified with $9 + 0i$.

Operations on complex numbers.

$$\text{real part } \operatorname{Re}(x + yi) := x$$

$$\text{imaginary part } \operatorname{Im}(x + yi) := y \quad (\text{Note: It is } y, \text{ not } yi, \text{ so } \operatorname{Im}(x + yi) \text{ is real})$$

$$\text{complex conjugate } \overline{x + yi} := x - yi \quad (\text{negate the imaginary component})$$

e.g. If $z = 3 + 4i$, then $\bar{z} = 3 - 4i$.

Properties of complex conjugation:

$$\begin{aligned} (1) \quad \overline{\overline{z}} &= z. \\ (2) \quad \overline{z + w} &= \overline{z} + \overline{w}. \\ (3) \quad \overline{z \cdot w} &= \overline{z} \cdot \overline{w}. \end{aligned} \quad \text{Re } z = \frac{z + \overline{z}}{2}, \quad \text{Im } z = \frac{z - \overline{z}}{2i}.$$

Example. Solve $x^2 - 2x + 3 = 0$.

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(3)}}{2(1)} = \frac{2 \pm \sqrt{-8}}{2} = \frac{2 \pm 2\sqrt{-2}}{2} = 1 \pm \sqrt{2}i.$$

Example: Consider the equation:

$$x^2 - 6x + 25 = 0 \quad \dots \quad (i)$$

$$x^2 - 6x + 25 = 0$$

$$(x - 3)^2 + 16 = 0$$

$$(x - 3)^2 = -16$$

$$(x - 3) = \pm \sqrt{-16} = \pm \sqrt{16 \cdot (-1)}.$$

$$x = 3 \pm 4i$$

The roots of the above equation (i) are $3 + 4i$ and $3 - 4i$.

Equality of Two Complex Numbers

Given that $z_1 = a + bi$ and $z_2 = c + di$

where $z_1, z_2 \in C$.

Two complex numbers are equal iff the real parts and the imaginary parts are respectively equal.

So, if $z_1 = z_2$, then $a = c$ and $b = d$.

Example 1. Find x, y if $(3 + 4i)^2 - 2(x - iy) = x + iy$.

$$\begin{aligned} \text{Left hand side (LHS)} &= 9 - 16 + 24i - 2x + i2y \\ &= -7 - 2x + i(24 + 2y) \end{aligned}$$

$$\begin{aligned} \therefore -7 - 2x &= x & \& \quad 24 + 2y = y \\ 3x &= -7 & y &= -24 \\ x &= -\frac{7}{3} \end{aligned}$$

Complex Arithmetic.

Addition/Subtraction.

If $z_1 = a + bi$ and $z_2 = c + di$ are two complex numbers, then

$$z_1 \pm z_2 = (a + c) \pm (b + d)i$$

Example 1. $(2 + 3i) + (4 + i) = 6 + 4i$.

Example 2. $(8 - 3i) - (-2 + 4i) = 10 - 7i$.

Try it Now

1. Subtract $2 + 5i$ from $3 - 4i$.

Multiplication/Division.

If $z_1 = a + bi$ and $z_2 = c + di$ are two complex numbers, and k is a constant, then

1. $z_1 \cdot z_2 = (a + bi) \cdot (c + di)$

2. $kz_1 = ka + kbi$

Example 1. $(2 + 3i)(1 + 2i) = 2 + 4i + 3i - 6 = -4 + 7i$

Example 2. $(3 - 2i)(3 + 2i) = 9 - (2i)^2 = 9 + 4 = 13$

Try it Now

2. Multiply $3 - 4i$ and $2 + 3i$.

Division:

If we are dividing with a complex number, the denominator must be converted to a real number. In order to do that, multiply both the denominator and numerator by complex conjugate of the denominator ,i.e.

To divide z by w , multiply z/w by $\bar{w}/\bar{w}$ so that the denominator becomes real; for example,

$$\frac{2 + 3i}{1 - 5i} = \frac{2 + 3i}{1 - 5i} \cdot \frac{1 + 5i}{1 + 5i} = \frac{2 + 13i + 15i^2}{1 - 25i^2} = \frac{-13 + 13i}{26} = -\frac{1}{2} + \frac{1}{2}i.$$

We know that

$$i^2 = -1$$

$$i^5 = (i^2)^2 \cdot i = 1 \cdot i = i$$

$$i^3 = i^2 \cdot i = -1 \cdot i = -i$$

$$i^6 = (i^2)^3 = (-1)^3 = -1$$

$$i^4 = (i^2)^2 = (-1)^2 = 1$$

$$i^7 = (i^2)^3 (i) = -i$$

$$i^8 = (i^2)^4 = 1$$

Example: Find the value of $1 + i^{10} + i^{20} + i^{30}$

Solution: $1 + i^{10} + i^{20} + i^{30}$

$$= 1 + (i^2)^5 + (i^2)^{10} + (i^2)^{15}$$

$$= 1 + (-1)^5 + (-1)^{10} + (-1)^{15}$$

$$= 1 + (-1) + 1 + (-1)$$

$$= 1 - 1 + 1 - 1$$

$$= 0$$

Homework

Find the values of :

$$1) (1 + 2i)^3 \quad 2) \frac{5}{-3 + 4i} \quad 3) \left(\frac{2 + i}{3 - 2i} \right)^2$$

The Argand Diagram.

Two methods: i. $P(x, y)$ the point P on the (x, y) -plane

ii. Vector $\overrightarrow{OP}$

x -axis is called the *real axis*.

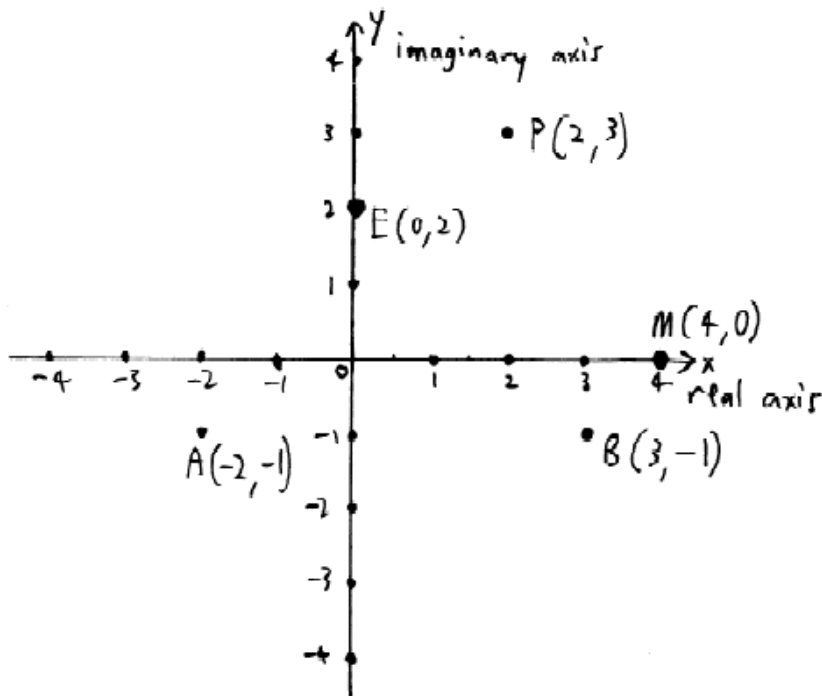
y -axis is called the *imaginary axis*.

(Note: Ordered pairs:- eg. $2 + i = (2, 1)$

for $2 + i = x + iy$ on (x, y) -plane)

Example: Plot the following on the Argand diagram:

$$P = 2 + 3i; B = 3 - i; A = -2 - i; M = 4; E = 2i$$



Polar form of a complex number. Given a nonzero complex number $z = x + yi$, we can express the point (x, y) in polar coordinates r and θ :

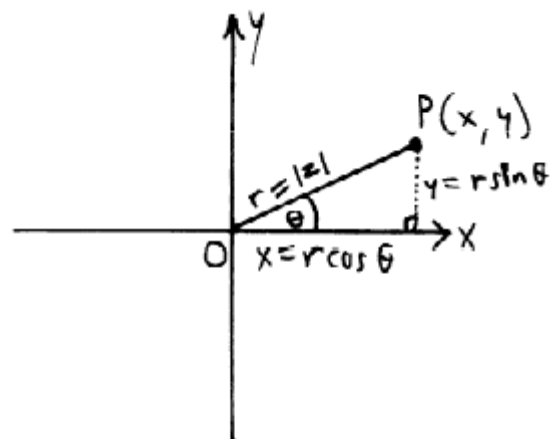
$$\begin{aligned} z &= x + iy \\ &= r \cos \theta + ir \sin \theta \\ &= r(\cos \theta + i \sin \theta) \end{aligned}$$

Modulus (Distance OP)

denoted by r , $\text{mod } z$, $|z|$, $|x + iy|$
by Pythagoras, $r^2 = x^2 + y^2$

$$r = \sqrt{x^2 + y^2}$$

$$r = |z| = |x + iy| = \sqrt{x^2 + y^2}.$$





Argument (angle θ)

denoted by θ , $\arg z$, $\arg(x + iy)$ [or $\text{amp } z$, $\text{amp } (x + iy)$ {amplitude}]

by definition, $-180^\circ < \theta \leq 180^\circ$

For $x \neq 0$, $\tan \theta = \frac{y}{x}$.

The mod-arg form of a complex number

$$z = x + iy$$

$$= r(\cos \theta + i \sin \theta)$$

$$(= r \text{ cis } \theta).$$

Radian measure (or circular measure)

$$\text{eg. } 360^\circ = 2\pi \text{ radians} = 2\pi \text{ rad} = 2\pi^c = 2\pi$$

$$180^\circ = \pi$$

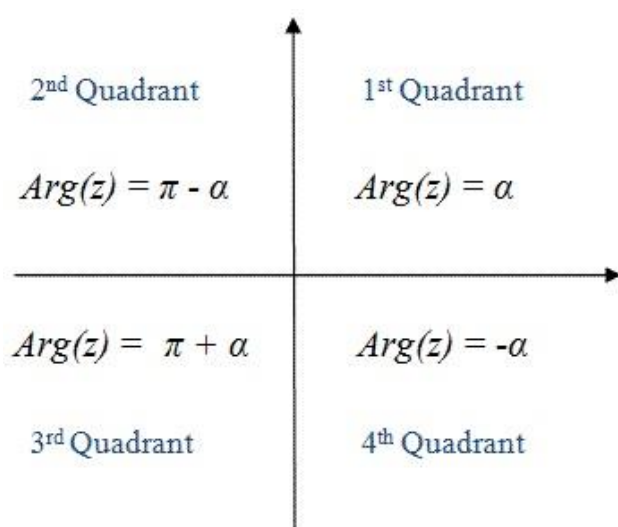
$$90^\circ = \frac{\pi}{2}$$

$$60^\circ = \frac{\pi}{3}$$

$$45^\circ = \frac{\pi}{4}$$

$$30^\circ = \frac{\pi}{6}$$

Note:



Quadrant	Sign of x & y	Arg(z)
I	$x, y > 0$	$\tan^{-1} \frac{y}{x}$
II	$x < 0, y > 0$	$\pi - \tan^{-1} \left \frac{y}{x} \right $
III	$x, y < 0$	$\pi + \tan^{-1} \left \frac{y}{x} \right $
IV	$x > 0, y < 0$	$-\tan^{-1} \left \frac{y}{x} \right $

Examples. Express the following in mod-arg form:-

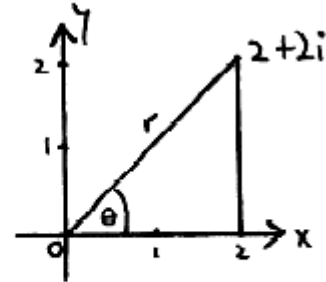
- (a) $2 + 2i$; (b) $2 + 5i$; (c) $-1 + \sqrt{3}i$; (d) $3i$; (e) $1 - 3i$

(a) $2 + 2i$

$$r = \sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$$

$$\& \tan \theta = \frac{2}{2} = 1 \& \therefore \theta = \frac{\pi}{4}$$

$$\& \therefore 2 + 2i = 2\sqrt{2} \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right). \quad \square$$

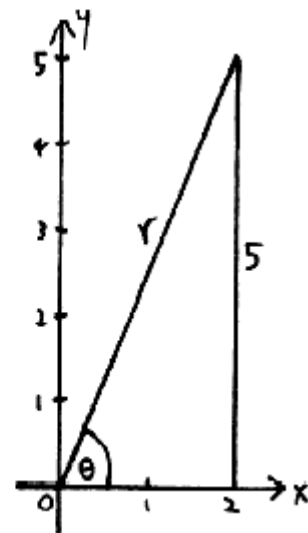


(b) $2 + 5i$

$$r = \sqrt{2^2 + 5^2} = \sqrt{29}$$

$$\& \tan \theta = \frac{5}{2} \& \therefore \theta = \tan^{-1} \frac{5}{2} \approx 68^\circ 12'$$

$$\& \therefore 2 + 5i = \sqrt{29} \left(\cos \left(\tan^{-1} \frac{5}{2} \right) + i \sin \left(\tan^{-1} \frac{5}{2} \right) \right) \\ \approx \sqrt{29} \left(\cos 68^\circ 12' + i \sin 68^\circ 12' \right) \quad \square$$

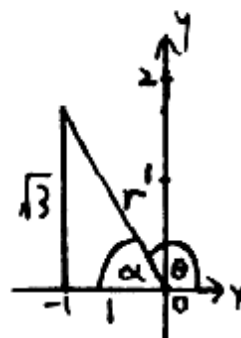


(c) $-1 + \sqrt{3}i$

$$r = \sqrt{1^2 + 3} = \sqrt{4} = 2$$

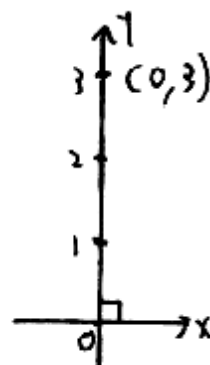
$$\tan \alpha = \frac{\sqrt{3}}{1} \text{ \& \therefore } \alpha = \frac{\pi}{3} \text{ \& \therefore } \theta = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

$$\text{\& \therefore } -1 + \sqrt{3}i = 2 \left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \right). \quad \square$$



(d) $3i$

By inspection, $3i = 3 \left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \right). \quad \square$



(e) $1 - 3i$

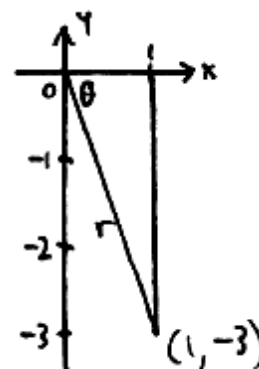
$$r = \sqrt{1^2 + 3^2} = \sqrt{10}$$

$$\tan(-\theta) = 3 \text{ \& \therefore } -\theta = \tan^{-1} 3 \text{ \& \therefore } \theta = -\tan^{-1} 3 \approx 71^\circ 34'$$

$$\text{\& \therefore } 1 - 3i = \sqrt{10} \left(\cos(-\tan^{-1} 3) + i \sin(-\tan^{-1} 3) \right)$$

$$= \sqrt{10} \left(\cos(\tan^{-1} 3) - i \sin(\tan^{-1} 3) \right)$$

$$\approx \sqrt{10} \left(\cos 71^\circ 34' - i \sin 71^\circ 34' \right) \quad \square$$



Try it Now

3. Write $-\sqrt{3} - i$ in polar form.

The n -th Power Of A Complex Number

DeMoivre's Theorem

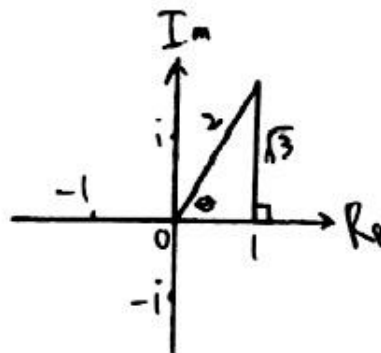
If $z = r(\cos(\theta) + i \sin(\theta))$, then for any integer n , $z^n = r^n (\cos(n\theta) + i \sin(n\theta))$

Example: Express in the form $x + iy$:

(a) $(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2})^6$ (b) $(1 + \sqrt{3})^{10}$.

$$\begin{aligned} \text{(a)} \quad (\cos \frac{\pi}{2} + i \sin \frac{\pi}{2})^6 &= \cos \frac{6\pi}{2} + i \sin \frac{6\pi}{2} \\ &= \cos 3\pi + i \sin 3\pi \\ &= -1 + 0i \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad (1 + \sqrt{3})^{10} &= (2(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}))^{10} \\ &= 2^{10}(\cos \frac{10\pi}{3} + i \sin \frac{10\pi}{3}) \\ &= 2^{10}(-\frac{1}{2} - i\frac{\sqrt{3}}{2}) \\ &= -512 - 512i\sqrt{3} \end{aligned}$$



Try it Now

4. Evaluate $(\sqrt{3} + i)^6$ using polar form.

Properties of $\text{cis}\theta$ [$\text{cis}\theta = \cos\theta + i \sin\theta$]

1. $\text{cis}\theta_1 \cdot \text{cis}\theta_2 = \text{cis}(\theta_1 + \theta_2)$

$$\begin{aligned} \text{cis}\theta_1 \cdot \text{cis}\theta_2 &= (\cos\theta_1 + i \sin\theta_1)(\cos\theta_2 + i \sin\theta_2) \\ &= \cos\theta_1 \cdot \cos\theta_2 + i \sin\theta_2 \cos\theta_1 + i \sin\theta_1 \cos\theta_2 - \sin\theta_1 \cdot \sin\theta_2 \\ &= (\cos\theta_1 \cdot \cos\theta_2 - \sin\theta_1 \cdot \sin\theta_2) + i(\sin\theta_2 \cos\theta_1 + \sin\theta_1 \cos\theta_2) \\ &= \cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2) = \text{cis}(\theta_1 + \theta_2) \end{aligned}$$

$$2. (\text{cis } \theta)^{-1} = \text{cis}(-\theta)$$

$$3. \frac{\text{cis } \theta_1}{\text{cis } \theta_2} = \text{cis}(\theta_1 - \theta_2)$$

These properties lead to an interacting geometry interpretation of the product, quotient of two complex numbers, in terms of vectors that represent them.

Product

$$z_1 = r_1(\cos \theta_1 + i \sin \theta_1) = r_1 \cdot \text{cis } \theta_1$$

$$z_2 = r_2(\cos \theta_2 + i \sin \theta_2) = r_2 \cdot \text{cis } \theta_2$$

$$z = z_1 \cdot z_2 = (r_1 \cdot \text{cis } \theta_1)(r_2 \cdot \text{cis } \theta_2) = r_1 \cdot r_2 \cdot \text{cis}(\theta_1 + \theta_2)$$

Example:

if $z_1 = 1 + i$ and $z_2 = \sqrt{3} - i$ find $z_1 \cdot z_2$

$$z_1 = \sqrt{2} \text{cis}(45) \quad , z_2 = 2 \text{cis}(-30)$$

$$z_1 \cdot z_2 = (\sqrt{2} \text{cis } 45)(2 \text{cis}(-30)) = 2\sqrt{2} \text{cis}(45 - 30) = 2\sqrt{2}(\cos 15 + i \sin 15) = 2.73 + i0.73$$

Division

$$z_1 = r_1(\cos \theta_1 + i \sin \theta_1) = r_1 \cdot \text{cis } \theta_1$$

$$z_2 = r_2(\cos \theta_2 + i \sin \theta_2) = r_2 \cdot \text{cis } \theta_2$$

$$\frac{z_1}{z_2} = \frac{r_1 \cdot \text{cis } \theta_1}{r_2 \cdot \text{cis } \theta_2} = \frac{r_1}{r_2} \text{cis}(\theta_1 - \theta_2)$$

Example:

if $z_1 = 1 + i$ and $z_2 = \sqrt{3} - i$ obtain $\frac{z_1}{z_2}$ for the previous complex numbers.

$$\frac{z_1}{z_2} = \frac{r_1 \cdot \text{cis } \theta_1}{r_2 \cdot \text{cis } \theta_2} = \frac{r_1}{r_2} \text{cis}(\theta_1 - \theta_2)$$

$$= \frac{\sqrt{2}}{2} \text{cis}(45 - (-30)) = \frac{1}{\sqrt{2}} \text{cis}(75) = 0.707(\cos 75 + i \sin 75) = 0.18 + i0.68$$

Power

Demoivers theory which states that :

$$(cis\theta)^n = cis(n\theta)$$

$$z = (cis\theta)^n = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

$$(cis\theta)^n = cis\theta.cis\theta.cis\theta.cis\theta.cis\theta..... = cis(\theta + \theta + \theta + \theta + \theta + \theta + \theta + ...) = cis(n\theta)$$

$$= \cos n\theta + i\sin n\theta$$

Example:

Obtain $cis3\theta$ in term of $\sin\theta$, $\cos\theta$.

$$cis3\theta = (\cos\theta + i\sin\theta)^3 = \cos3\theta + i\sin3\theta$$

$$= \cos^3\theta + 3\cos^2\theta(i\sin\theta) + 3\cos\theta(i\sin\theta)^2 + (i\sin\theta)^3 = \cos^3\theta - 3\sin^2\theta\cos\theta + i(3\sin\theta\cos^2\theta - \sin^3\theta)$$

$$\therefore \cos3\theta = \cos^3\theta - 3\sin^2\theta\cos\theta$$

$$\therefore \sin3\theta = 3\sin\theta\cos^2\theta - \sin^3\theta$$

De Moivre's Theorem to Prove Trigonometric Identities

De-Moivre's theorem can be used to prove some trigonometric identities. (with the help of Binomial theorem or Pascal triangle.)

Pascal's Triangle:

1	$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3,$
1 1	$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4,$
1 2 1	$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5,$
1 3 3 1	$(x+y)^6 = x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6,$
1 4 6 4 1	$(x+y)^7 = x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + y^7.$
1 5 10 10 5 1	
1 6 15 20 15 6 1	
.....	

Example: Prove that

$$\cos5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta \text{ and}$$

$$\sin5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta.$$

Solution: The idea is to write $(\cos\theta + i\sin\theta)^5$ in two different ways. We use both the Pascal triangle and De Moivre's theorem, and compare the results.

$$(\cos\theta + i\sin\theta)^5 = \cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - 10i\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta$$

$$= (\cos^5 \theta - 10 \cos^3 \theta \sin^2 \theta + 5 \cos \theta \sin^4 \theta) + i(5 \cos^4 \theta \sin \theta - 10 \cos^2 \theta \sin^3 \theta + \sin^5 \theta).$$

Also, by De Moivre's Theorem, we have $(\cos \theta + i \sin \theta)^5 = \cos 5\theta + i \sin 5\theta$. and so
 $\cos 5\theta + i \sin 5\theta = (\cos^5 \theta - 10 \cos^3 \theta \sin^2 \theta + 5 \cos \theta \sin^4 \theta) + i(5 \cos^4 \theta \sin \theta - 10 \cos^2 \theta \sin^3 \theta + \sin^5 \theta).$

Equating the real parts gives

$$\begin{aligned} \cos 5\theta &= \cos^5 \theta - 10 \cos^3 \theta \sin^2 \theta + 5 \cos \theta \sin^4 \theta. \\ &= \cos^5 \theta - 10 \cos^3 \theta (1 - \cos^2 \theta) + 5 \cos \theta (1 - \cos^2 \theta)^2 \\ &= \cos^5 \theta - 10 \cos^3 \theta + 10 \cos^5 \theta + 5 \cos \theta - 10 \cos^3 \theta + 5 \cos^5 \theta. \\ &= 16 \cos^5 \theta - 20 \cos^3 \theta + 5 \cos \theta. \text{ (proved)} \end{aligned}$$

Equating the imaginary parts gives

$$\begin{aligned} \sin 5\theta &= 5 \cos^4 \theta \sin \theta - 10 \cos^2 \theta \sin^3 \theta + \sin^5 \theta \\ &= 5(1 - \sin^2 \theta)^2 \sin \theta - 10(1 - \sin^2 \theta) \sin^3 \theta + \sin^5 \theta \\ &= 5(1 - 2 \sin^2 \theta + \sin^4 \theta) \sin \theta - 10 \sin^3 \theta + 10 \sin^5 \theta + \sin^5 \theta \\ &= 5 \sin \theta - 10 \sin^3 \theta + 5 \sin^5 \theta - 10 \sin^3 \theta + 10 \sin^5 \theta + \sin^5 \theta \\ &= 16 \sin^5 \theta - 20 \sin^3 \theta + 5 \sin \theta \text{ (proved).} \end{aligned}$$

The n -th Roots of a Complex Number

A complex number w is a n -th root of the complex number z if $w^n = z$ or w

$$\begin{aligned} &= z^{\frac{1}{n}}. \\ &= [r(\cos \theta + i \sin \theta)]^{\frac{1}{n}}, \\ &= r^{1/n} \left(\cos \frac{\theta + 2k\pi}{n} + i \sin \frac{\theta + 2k\pi}{n} \right) \text{ for } k = 0, 1, 2, \dots, n-1 \end{aligned}$$

Substituting $k = 0, 1, 2, \dots, n-1$ yields the n th roots of the given complex number.

Example:

Find the three roots of -32.

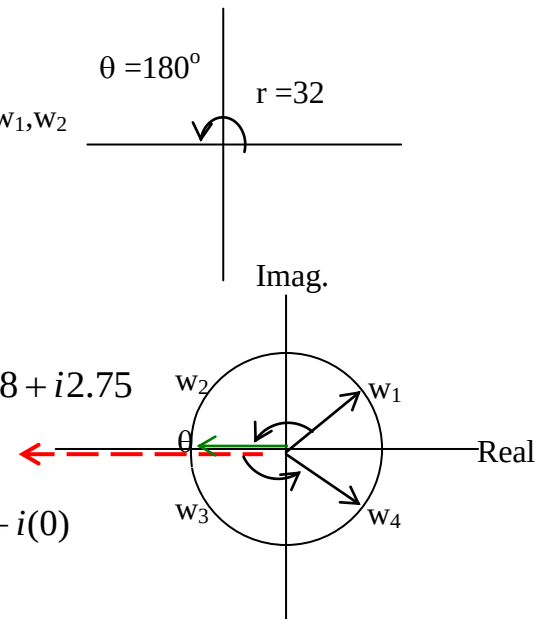
$$w_0, w_1, w_2, n=3, k=0,1,2 \quad w_0, w_1, w_2$$

$$w = r^{1/n} \left[\cos\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right) + i \sin\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right) \right]$$

$$w_0 = (32)^{1/3} \left[\cos\left(\frac{180}{3} + \frac{2\pi(0)}{3}\right) + i \sin\left(\frac{180}{3} + \frac{2\pi(0)}{3}\right) \right] = 1.58 + i2.75$$

$$w_1 = (32)^{1/3} \left[\cos\left(\frac{180}{3} + \frac{2\pi(1)}{3}\right) + i \sin\left(\frac{180}{3} + \frac{2\pi(1)}{3}\right) \right] = -3.17 + i(0)$$

$$w_2 = (32)^{1/3} \left[\cos\left(\frac{180}{3} + \frac{2\pi(2)}{3}\right) + i \sin\left(\frac{180}{3} + \frac{2\pi(2)}{3}\right) \right] = 1.58 - i2.75$$



- ❖ أول جذر يرسم بزاوية مقدارها (θ/n) بالاتجاه الموجب. إذا حسب مثالنا اعلاه $[(\theta/n=180/3=60^\circ)]$
- ❖ نرسم الزاوية بين جذر واخر $(360/n)$ بالاتجاه الموجب. أي $[(360/3=120^\circ)]$

Example:

$$z^4 = 1 + i\sqrt{3}$$

$$= 2 \left(\frac{1}{2} + i\frac{\sqrt{3}}{2} \right)$$

$$= 2 \left[\cos\left(\frac{\pi}{3}\right) + i \sin\left(\frac{\pi}{3}\right) \right]$$

$$= 2e^{i\frac{\pi}{3}}$$

$$z_0 = 2^{\frac{1}{4}} e^{i\frac{\pi}{12}} = 2^{\frac{1}{4}} \left[\cos\left(\frac{\pi}{12}\right) + i \sin\left(\frac{\pi}{12}\right) \right]$$

$$z_1 = 2^{\frac{1}{4}} e^{i\frac{7\pi}{12}} = 2^{\frac{1}{4}} \left[\cos\left(\frac{7\pi}{12}\right) + i \sin\left(\frac{7\pi}{12}\right) \right]$$

Hence: $z = 2^{\frac{1}{4}} e^{i\left(\frac{\pi}{12} + \frac{2k\pi}{4}\right)}$ (divide argument by four)

$$z_2 = 2^{\frac{1}{4}} e^{i\frac{13\pi}{12}} = 2^{\frac{1}{4}} \left[\cos\left(\frac{13\pi}{12}\right) + i \sin\left(\frac{13\pi}{12}\right) \right] \quad \text{This has four distinct roots.}$$

$$z_3 = 2^{\frac{1}{4}} e^{i\frac{19\pi}{12}} = 2^{\frac{1}{4}} \left[\cos\left(\frac{19\pi}{12}\right) + i \sin\left(\frac{19\pi}{12}\right) \right]$$

Polar Form of a Complex Number and Euler's Formula

The **polar form of a complex number** is $z = r \cos(\theta) + ir \sin(\theta)$.

An alternate form, which will be the primary one used, is $z = re^{i\theta}$

Euler's Formula states $re^{i\theta} = r \cos(\theta) + ir \sin(\theta)$

Similar to plotting a point in the polar coordinate system we need r and θ to find the polar form of a complex number.

The n -th Power Of A Complex Number

We know that a complex number can be express as $z = re^{i\theta}$, then

$$z^2 = r^2 e^{i2\theta}$$

$$z^3 = r^3 e^{i3\theta}$$

$$z^4 = r^4 e^{i4\theta}$$

:

$$z^n = r^n e^{in\theta}$$

The n -th Roots Of A Complex Number

The n -th roots of a complex number can be found using the Euler's formula.

Note that:

$$z = re^{i(\theta+2k\pi)}$$

Then,

$$z^{\frac{1}{2}} = r^{\frac{1}{2}} e^{\left(\frac{\theta+2k\pi}{2}\right)i}, \quad k = 0,1$$

$$z^{\frac{1}{3}} = r^{\frac{1}{3}} e^{\left(\frac{\theta+2k\pi}{3}\right)i}, \quad k = 0,1,2$$

$$z^{\frac{1}{n}} = r^{\frac{1}{n}} e^{\left(\frac{\theta+2k\pi}{n}\right)i}, \quad k = 0,1,2,\dots,n-1$$



Functions of a Complex Variable

If, to each value of a complex number z in some region of the complex plane there corresponds a unique value of another complex variable w , then w is a function of z , i.e. a function of complex variable, written $w = f(z)$. Let u and v be the real and imaginary parts of w , then since $z = x + iy$, u and v will in general be functions of x and y and we write $w = u(x,y) + iv(x,y)$

Example:

$w = z^2 - z$ is defined for all $z \in \mathbb{C}$.

$$\begin{aligned}\text{Since } w &= (x + iy)^2 - (x + iy) \\ &= (x^2 - y^2 - x) + i(2xy - y)\end{aligned}$$

we have $u(x,y) = x^2 - y^2 - x$

$$v(x,y) = 2xy - y$$

For example, $f(3+2i) = 2+10i$

Example:

$$\begin{aligned}w &= 3\bar{z} + z \\ &= 3(x - iy) + (x + iy) \\ &= 4x - 2iy \\ \therefore u(x,y) &= 4x, \quad v(x,y) = -2y\end{aligned}$$

and $f(4 - i) = 16 + 2i$

Relationship between circular and hyperbolic functions.

Euler's formula provides the theoretical link between circular and hyperbolic functions. Since

$$e^{i\theta} = \cos \theta + i \sin \theta \quad \text{and} \quad e^{-i\theta} = \cos \theta - i \sin \theta$$

we deduce that:

$$\boxed{\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}} \quad \text{and} \quad \boxed{\sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}}. \quad (1)$$

We defined the hyperbolic function by:

$$\boxed{\cosh x = \frac{e^x + e^{-x}}{2}} \quad \text{and} \quad \boxed{\sinh x = \frac{e^x - e^{-x}}{2}} \quad (2)$$

Comparing (1) and (2), we have

$$\cosh ix = \frac{e^{ix} + e^{-ix}}{2} = \cos x$$

$$\sinh ix = \frac{e^{ix} - e^{-ix}}{2i} = i \sin x$$

so that

$$\tanh ix = i \tan x.$$

Also,

$$\cos ix = \frac{e^{i^2x} + e^{-i^2x}}{2} = \frac{e^{-x} + e^x}{2} = \cosh x \quad (3)$$

$$\sin ix = \frac{e^{i^2x} - e^{-i^2x}}{2i} = \frac{e^{-x} - e^x}{2} = \sinh x$$

so that : $\tan ix = i \tanh x$.

Using these results, we can evaluate functions such as $\sin z$, $\cos z$, $\tan z$, $\sinh z$, $\cosh z$ and $\tanh z$.

For example, to evaluate

$$\cos z = \cos(x + iy)$$

we use the identity

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

and obtain

$$\cos z = \cos x \cos iy - \sin x \sin iy$$

Using results in (3), this gives

$$\cos z = \cos x \cosh y - i \sin x \sinh y.$$

Homework: Find the values of

a) $\sinh(3 + 4i)$

b) $\tan\left(\frac{\pi}{4} - 3i\right)$

c) $\sin\left(\frac{\pi}{4}(1 + i)\right)$

(Ans: a) -6.548-7.619i

b) 0.005-1.0i

c) 0.9366+0.6142i)



Cauchy-Riemann equations:

If $f(z)$ any function of complex variable

$$\text{and } \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

These conditions are known as the **Cauchy-Riemann equations**.

Examples:

(i) $f(z) = z^2 = x^2 - y^2 + 2ixy$
 $\therefore u = x^2 - y^2 \quad v = 2xy$

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial v}{\partial y} = 2x$$

$$\frac{\partial u}{\partial y} = -2y, \quad \frac{\partial v}{\partial x} = 2y$$

Both equations are satisfied (C-R)eq.

(ii) $f(z) = \bar{z} = x - iy$
 $u = x \quad v = -y$

$$\frac{\partial u}{\partial x} = 1 \quad \frac{\partial v}{\partial y} = -1$$

$$\frac{\partial u}{\partial y} = 0 \quad \frac{\partial v}{\partial x} = 0$$

The first C-R equation is not satisfied

(iii) $f(z) = z\bar{z} = |z|^2 = x^2 + y^2$

$$u = x^2 + y^2 \quad v = 0$$

$$\frac{\partial u}{\partial x} = 2x \quad \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial u}{\partial y} = 2y \quad \frac{\partial v}{\partial x} = 0$$

The C-R equations are only satisfied at $z = 0$

Partial Derivatives

Let $f(x, y)$ be a function of two variables, then f_x is defined as the derivative of the function $f(x, y)$, where y is considered a constant. It is called partial derivative of f with respect to x . The partial derivative with respect to y is defined similarly.

For a function of more than two independent variables, the same method applies. Its partial derivative with respect to x , can be obtained by differentiating it with respect to x , using all the usual rules of differentiation. Except that all the other independent variables, whenever and wherever they occur in the expression of f , are treated as constants.

$f_x \equiv \frac{\partial f}{\partial x}$ means partial derivative of f with respect to x

$f_y \equiv \frac{\partial f}{\partial y}$ means partial derivative of f with respect to y

Example:

Find f_x and f_y for 1. $f(x, y) = x^2 y^4$ 2. $f(x, y) = x^3 + y^2$

1. $f_x = 2xy^4$ and $f_y = 4x^2 y^3$

2. $f_x = 3x^2$ and $f_y = 2y$

Example:

Find $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ for $w(x, y) = x^2 \cos(y)$

$$\frac{\partial w}{\partial x} = 2x \cos(y) \quad , \quad \frac{\partial w}{\partial y} = -x^2 \sin(y)$$

Example:

Find h_s and h_t for $h(s, t) = t \ln(4s^2 + 1) + t^2 \tan^{-1}(2s)$

$$h_s = \frac{8st}{4s^2 + 1} + \frac{2t^2}{1 + 4s^2} = \frac{8st + 2t^2}{4s^2 + 1}$$

$$h_t = \ln(4s^2 + 1) + 2t \tan^{-1}(2s)$$

Second order partial derivatives

Let $f(x, y)$ be a function of x and y , then the second partial derivative of f with respect to x is f_{xx} , the second partial derivative of f with respect to y is f_{yy} , the second partial derivative of f with respect to y and then with respect to x is f_{xy} and the second partial derivative of f with respect to x and then with respect to y is f_{yx} .

$$\left(f_{xx} \equiv \frac{\partial^2 f}{\partial x^2}, f_{yy} \equiv \frac{\partial^2 f}{\partial y^2}, f_{xy} \equiv \frac{\partial^2 f}{\partial x \partial y} \text{ and } f_{yx} \equiv \frac{\partial^2 f}{\partial y \partial x} \right)$$

and f_{yx} are called mixed partial derivatives and $f_{xy} = f_{yx}$.

Example:

Find f_{xx} , f_{xy} , f_{yx} and f_{yy} for $f(x, y) = x^2 + 3xy - y^4$

$$f_x = 2x + 3y \Rightarrow f_{xx} = 2 \quad \text{and} \quad f_{yx} = 3$$

$$f_y = 3x - 4y^3 \Rightarrow f_{yy} = -12y^2 \quad \text{and} \quad f_{xy} = 3$$

Chain rule for partial derivatives

If $z = f(x, y)$ and $x = x(u, v)$, $y = y(u, v)$ then z is a function of u and v and

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \times \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \times \frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \times \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \times \frac{\partial y}{\partial v}$$

the chain for the function $f(x, y, z)$.

If x , y and z are all functions of a single variable t , then f can be considered as a function of t and

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + \frac{\partial f}{\partial z} \frac{dz}{dt}$$

Example:

If $z = x^2 + y^2$, $x = 2u + v$, $y = 2v - u$.

Find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ as a functions of u and v

$$\frac{\partial z}{\partial x} = 2x = 4u + 2v \quad \text{and} \quad \frac{\partial z}{\partial y} = 2y = 4v - 2u$$

$$\frac{\partial x}{\partial u} = 2 \quad \text{and} \quad \frac{\partial y}{\partial u} = -1$$

$$\frac{\partial x}{\partial v} = 1 \quad \text{and} \quad \frac{\partial y}{\partial v} = 2$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \times \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \times \frac{\partial y}{\partial u}$$

$$= (4u + 2v) \times 2 + (4v - 2u) \times (-1) = 8u + 4v - 4v + 2u = 10u$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \times \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \times \frac{\partial y}{\partial v}$$

$$= (4u + 2v) \times 1 + (4v - 2u) \times 2 = 4u + 2v + 8v - 4u = 10v$$