

Complex Conjugate

The complex conjugate of the complex number $Z = a + ib$ is defined to be $\bar{Z} = \overline{a + ib} = a - ib$

The properties of the conjugate

① $\overline{\bar{Z}} = Z$, $Z = a + ib \Rightarrow \bar{Z} = a - ib \Rightarrow \overline{\bar{Z}} = a + ib = Z$

if $Z = 3 + 4i \Rightarrow \bar{Z} = 3 - 4i \Rightarrow \overline{\bar{Z}} = 3 + 4i = Z$

② $\bar{Z} = Z \Leftrightarrow Z$ is real, $Z = a \Rightarrow \bar{Z} = a$

$Z = -4 \Rightarrow \bar{Z} = -4$

③ $|Z| = |\bar{Z}|$

④ $|Z|^2 = Z \cdot \bar{Z}$

⑤ $\frac{Z + \bar{Z}}{2} = \operatorname{Re}(Z) = a$

$Z = 3 + 4i \Rightarrow \frac{Z + \bar{Z}}{2} = \frac{(3 + 4i) + (3 - 4i)}{2} = \frac{6}{2} = 3 = \operatorname{Re}(Z)$

⑥ $\frac{Z - \bar{Z}}{2i} = \operatorname{Im}(Z) = b$

$\frac{(3 + 4i) - (3 - 4i)}{2i} = \frac{8i}{2i} = 4 = \operatorname{Im}(Z)$

⑦ a. $\overline{Z_1 + Z_2} = \bar{Z}_1 + \bar{Z}_2$

b. $\overline{Z_1 - Z_2} = \bar{Z}_1 - \bar{Z}_2$

c. $\overline{Z_1 \cdot Z_2} = \bar{Z}_1 \cdot \bar{Z}_2$

d. $\overline{\left(\frac{Z_1}{Z_2}\right)} = \frac{\bar{Z}_1}{\bar{Z}_2}$

Ex

① $(-3+i) + (2+3i)$

$$(-3-i) + (2+3i) = -1+2i$$

② $(\pi - \sqrt{7}i) + (5-2i) = (\pi + \sqrt{7}i) + (5-2i) = (\pi+5) + (\sqrt{7}-2)i$

③ $(2-i)(-1-i) \Rightarrow (2-i)(-1+i) = (-2+1) + i(1+2)$
 $= -1+3i$

④ $(\sqrt{2}-2i)(\sqrt{2}-2i) = \text{H.W.}$

5) $z=5$ then $\bar{z}=5$

6) $z=i$ then $\bar{z}=-i$

7) $z=(3i-5)$ then $\bar{z} = \overline{(-5+3i)} = -5-3i$

8) If $z_1=3+4i$, $z_2=1+2i$ find $\overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2$

$$\overline{z_1 \cdot z_2} = \overline{(3+4i)(1+2i)} = \overline{(3-8) + (4+6)i} = \overline{-5+10i}$$
$$= -5-10i$$

$$\bar{z}_1 \cdot \bar{z}_2 = \overline{(3+4i)} \cdot \overline{(1+2i)} = (3-4i) \cdot (1-2i) = (3-8) + (-4-6)i$$
$$= -5-10i$$

(5)

The Polar Form of Complex numbers

Def The magnitude or modulus or absolute value of complex number $Z = a + ib$ is $|Z| = |a + ib| = \sqrt{a^2 + b^2}$ and the angle θ between Ox and Z is called the phase or argument of Z written $\theta = \arg(Z)$

$$\theta = \tan^{-1} \frac{b}{a} \quad \text{or} \quad \tan \theta = \frac{b}{a}$$

Ex Find θ , $|Z|$ for

(1) $Z = 5 + 5i$ (2) $Z = -\sqrt{3} - i$

(3) $Z = -1$, $Z = i$, $Z = e$, $Z = -i$

Solu:

(1) $|Z| = r = \sqrt{a^2 + b^2} = \sqrt{25 + 25} = 5\sqrt{2}$

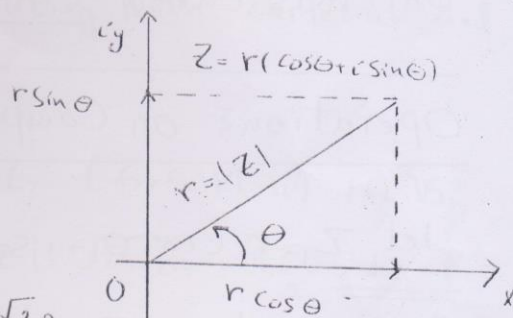
$$\theta = \tan^{-1} \frac{b}{a} = \tan^{-1} \frac{5}{5} = \tan^{-1} 1 = \frac{\pi}{4} \quad \Rightarrow r = 5\sqrt{2}, \theta = \frac{\pi}{4}$$

(2) $|Z| = \sqrt{(-\sqrt{3})^2 + (-1)^2} = 2 = r$

$$\theta = \tan^{-1} \frac{b}{a} = \tan^{-1} \frac{-1}{-\sqrt{3}} = \tan^{-1} \frac{1}{\sqrt{3}} = \frac{\pi}{6}$$

(3) $|Z| = \sqrt{(-1)^2 + 0} = 1 = r$

$$\theta = \tan^{-1} \frac{0}{-1} = 0 \Rightarrow \theta = \pi$$



$$(2) Z = -\sqrt{3} - i$$

$$\theta = \tan^{-1} \frac{b}{a}, |Z| = \sqrt{a^2 + b^2}$$

$$|Z| = \sqrt{(-\sqrt{3})^2 + (-1)^2} = \sqrt{3+1} = \sqrt{4} = 2 = r$$

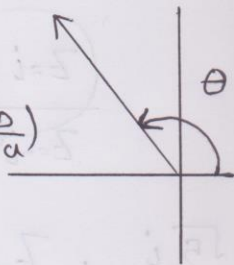
$$\theta = \tan^{-1} \frac{-1}{-\sqrt{3}} = \tan^{-1} \frac{1}{\sqrt{3}} = \frac{\pi}{6}$$

The following diagram show how to find " θ " depending on the quadrant

Second quadrant

$$Z = a + ib$$

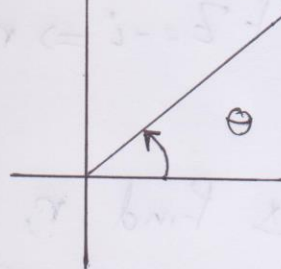
$$\theta = \pi - \tan^{-1} \left(\frac{b}{a} \right)$$



First quadrant

$$Z = a + ib$$

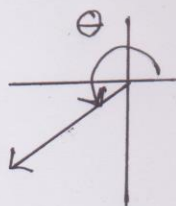
$$\theta = \tan^{-1} \left(\frac{b}{a} \right)$$



Third quadrant

$$Z = a + ib$$

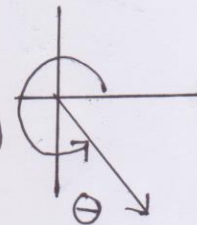
$$\theta = \pi + \tan^{-1} \left(\frac{b}{a} \right)$$



Fourth quad

$$Z = a + ib$$

$$\theta = 2\pi - \tan^{-1} \left(\frac{b}{a} \right)$$



$$\therefore \theta = \pi + \frac{\pi}{6} = \frac{6\pi + \pi}{6}$$

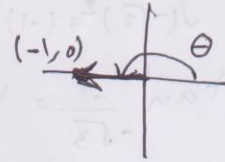
$$\theta = \frac{7\pi}{6}$$

$$Z = -1$$

$$(3) |Z| = \sqrt{(-1)^2 + 0^2} = 1 = r$$

$$\theta = \tan^{-1} \frac{b}{a} = \tan^{-1} \frac{0}{-1} = 0, \quad \theta = \underline{0} \text{ or } \pi$$

$$\theta = \pi$$



$$Z = 1 \Rightarrow r = 1, \quad \theta = \tan^{-1} \frac{0}{1} = 0 \Rightarrow \theta = 0$$

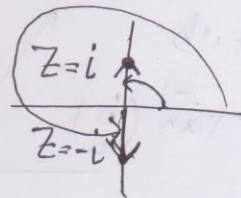
$$Z = i \Rightarrow Z = 0 + i \Rightarrow r = \sqrt{0^2 + 1^2} = 1$$

$$\theta = \tan^{-1} \frac{b}{a} = \tan^{-1} \frac{1}{0} \text{ (undefined)}$$

$$\Rightarrow \theta = \frac{\pi}{2}$$

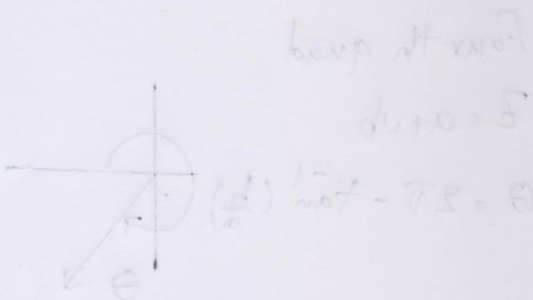
$$\theta = \frac{\pi}{2} \text{ or } \frac{3\pi}{2}$$

$$\text{if } Z = -i \Rightarrow r = 1, \quad \theta = \frac{3\pi}{2}$$



Ex Find r, θ if $Z_1 = -\sqrt{5}i$, $Z_2 = -\frac{3}{2}i + \frac{3}{2}$

(H.W.)



$$\frac{\pi + \pi}{2} = \frac{\pi}{2}, \quad \pi = \theta$$

$$\frac{\pi - \pi}{2} = \theta$$