

Double Integration

We shall show how to find a region bounded by two curves corresponding to the xy -plane using double integration.

In the figure below, we have the region (A) is bounded by the curves ($y = f_1(x)$, and $y = f_2(x)$) from above and below, and from left & right by ($x = a$, & $x = b$).

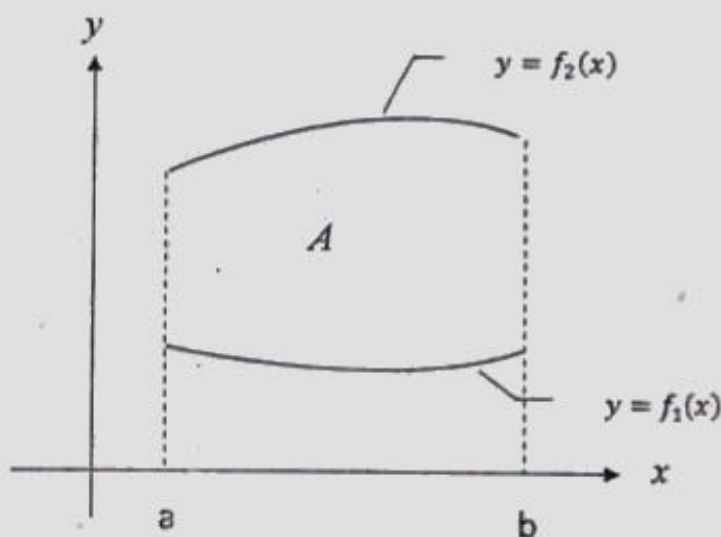


Figure - 1

Now we shall first define what mean by double integral of a function $f(x,y)$ of two variable x and y . Then specific applications follow at once by specializing the function $f(x,y)$ to be:-

(a) $f(x,y) = 1$

(b) $f(x,y) = y$

For the calculation of :-

(a) The area, or

(b) The moment of the area about the x -axis.

The notation

$$\iint_A f(x,y) dA \quad \text{--- (1)}$$

Mathematics - Multiple Integrals

Is used to denote the double integration over the region (A) of the function $f(x, y)$. Now we imagine the region (A) as covered by a grid of lines parallel to the x -axis and y -axis. These lines divide the plane into small pieces of area

$$\Delta A = \Delta x \Delta y = \Delta y \Delta x \quad \dots (2)$$

We focus our attention upon the elements ΔA lying inside the region and number them in some order as

$$\Delta A_1, \Delta A_2, \Delta A_3, \dots, \Delta A_n, \quad \dots (3)$$

Let (x_k, y_k) be any point lying in ΔA_k and form the sum

$$S_n = \sum_{k=1}^n F(x_k, y_k) \Delta A_k \quad \dots (4)$$

Finally in practice, the double integration in eq. (1) is evaluated by calculating one or another of the iterated integrals.

$$\iint_A F(x, y) dy dx \quad \text{or} \quad \iint_A F(x, y) dx dy \quad \dots (6)$$

Now we shall explain what we mean by iterated integral

$$\iint_A F(x, y) dy dx$$

It is the result of:-

- (a) Integrating $\int F(x, y) dy$ with respect to y (with x held fixed) and evaluating the resulting integral between the limits $y = f_1(x)$ & $y = f_2(x)$
- (b) Integrating the result of (a) with respect to x between the limits $x = a$ and $x = b$

That is we, start with innermost integral and perform successive integrations as follows:

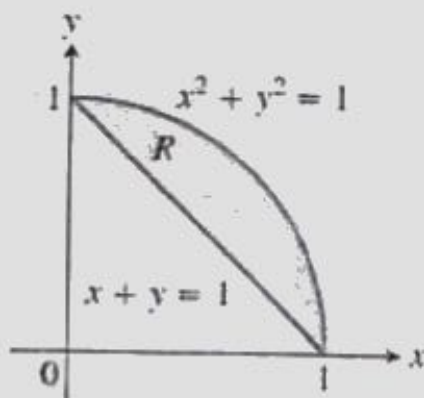
$$\iint_A F(x, y) dy dx = \int_a^b \left(\int_{f_1(x)}^{f_2(x)} F(x, y) dy \right) dx \quad \dots (7)$$

Finding limits of integration:-

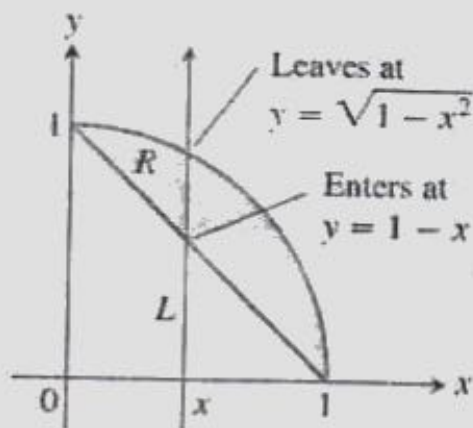
We now give a procedure for finding limits of integration that applied for many regions in the plane. Regions that are more complicated, and for which this procedure fails, can often be split up into pieces on which the procedure works.

When faced with evaluating $\iint_A f(x, y) dA$ integrating first with respect to y and then with respect to x , we will do the following:-

- 1) Sketch the region of integration and label the bounding curves.



- 2) Find the y -limits of integration, by imagine a vertical line (L) cutting through (R) in the direction of increasing y , marking the inter and leaves points of (L). these are limits of integration and usually functions of x (instead of constants).

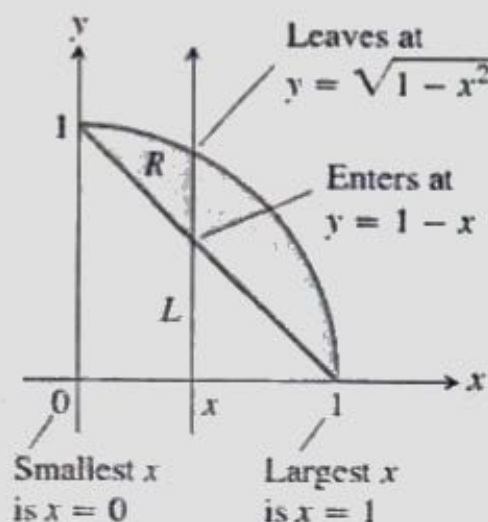


(12)

Mathematics - Multiple Integrals

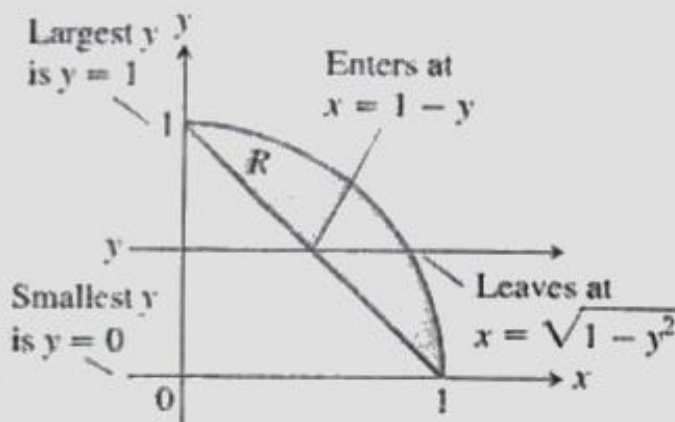
- 3) Find the *x*-limits of integration: Choose *x*-limits that include all the vertical lines through *R*. the integral shown here is

$$\iint_R f(x, y) dA = \int_{x=0}^{x=1} \int_{y=1-x}^{y=\sqrt{1-x^2}} f(x, y) dy dx$$



To evaluate the same double integral as in iterated with the order of integration reversed, use horizontal lines instead of vertical lines in step 2 and 3. The integral is :

$$\iint_R f(x, y) dA = \int_{y=0}^{y=1} \int_{x=1-y}^{x=\sqrt{1-y^2}} f(x, y) dx dy$$



Mathematics - Multiple Integrals

Properties of double integrals:-

Like single integral, double integrals of continuous function have algebraic properties that are useful in the computations and applications.

Properties of Double Integrals

If $f(x, y)$ and $g(x, y)$ are continuous, then

1. *Constant Multiple:* $\iint_R c f(x, y) dA = c \iint_R f(x, y) dA$ (any number c)

2. *Sum and Difference:*

$$\iint_R (f(x, y) \pm g(x, y)) dA = \iint_R f(x, y) dA \pm \iint_R g(x, y) dA$$

3. *Domination:*

(a) $\iint_R f(x, y) dA \geq 0$ if $f(x, y) \geq 0$ on R

(b) $\iint_R f(x, y) dA \geq \iint_R g(x, y) dA$ if $f(x, y) \geq g(x, y)$ on R

4. *Additivity:* $\iint_R f(x, y) dA = \iint_{R_1} f(x, y) dA + \iint_{R_2} f(x, y) dA$

if R is the union of two nonoverlapping regions R_1 and R_2 (Figure 15.7).

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Multiple Integration

"Double Integral"

Def: let $f(x,y)$ be a function of two variables defined on a region R : $a \leq x \leq b$, $c \leq y \leq d$ then the double integral (or Iterated Integral) of $f(x,y)$ over R is defined by

$$\iint_R f(x,y) dA = \int_a^b \int_c^d f(x,y) dy dx = \int_c^d \int_a^b f(x,y) dx dy$$

The goal here is to find the volume of a solid under the function $f(x,y)$ [and above the xy -plane]

$$\text{Volume} = V = \iint_R f(x,y) dA$$

Examples

① Evaluate $\int_0^1 \int_0^1 (x^2 + y^2) dx dy$

Solution

$$\begin{aligned} \int_0^1 \left[\int_0^1 (x^2 + y^2) dx \right] dy &= \int_0^1 \left[\frac{x^3}{3} + y^2 x \right]_0^1 dy \\ &= \int_0^1 \left(\frac{1}{3} + y^2 \right) dy = \left[\frac{1}{3} y + \frac{y^3}{3} \right]_0^1 = \frac{1}{3} + \frac{1}{3} = \frac{2}{3} \end{aligned}$$

$$R: 2 \leq x \leq 4 ; 3 \leq y \leq 6 \quad f(x,y) = 2$$

$$\begin{aligned} V &= \iint_R f(x,y) dA = \int_3^6 \int_2^4 2 dx dy = \int_3^6 [2x]_2^4 dy \\ &= \int_3^6 (8-4) dy = \int_3^6 4 dy = 4y \Big|_3^6 = 24 - 12 = 12 \end{aligned}$$

③ Find the volume of the solid which base in the xy -plane and its region $1 \leq x \leq 3$, $0 \leq y \leq 2$ while the top is bounded by $2x+y$

Solution

$$\begin{aligned} V &= \iint_R f(x,y) dA = \int_0^2 \int_1^3 e^{2x+y} dx dy \\ &= \int_0^2 \left[\int_1^3 e^{2x} \cdot e^y dx \right] dy = \int_0^2 e^y \left[\frac{1}{2} \int_1^3 e^{2x} dx \right] dy \\ &= \int_0^2 \frac{1}{2} e^y \left[e^{2x} \Big|_1^3 \right] dy = \int_0^2 \frac{1}{2} e^y (e^6 - e^2) dy = \frac{1}{2} (e^6 - e^2) \int_0^2 e^y dy \\ &= \frac{1}{2} (e^6 - e^2) \left[e^y \Big|_0^2 \right] = \frac{1}{2} (e^6 - e^2) (e^2 - 1) \end{aligned}$$

④ Find the double integral over the following region

(a) $\iint_R \frac{1}{(2x+3y)^2} dA \quad 0 \leq x \leq 1 ; 1 \leq y \leq 2$

(b) $\iint_R 6xy^2 dA \quad 2 \leq x \leq 4 ; 1 \leq y \leq 2 \quad \text{ans: } (84)$

(c) $\iint_R x e^{xy} dA \quad -1 \leq x \leq 2 ; 0 \leq y \leq 1 \quad \text{ans: } (e^2 - e^{-1} - 3)$

Solution (a)

$$\begin{aligned}
 \int_1^2 \int_0^1 (2x+3y)^{-2} dx dy &= \int_1^2 \frac{1}{2} \left[\int_0^1 2(2x+3y)^{-2} dx \right] dy \\
 &= \int_1^2 \frac{1}{2} \left[-(2x+3y)^{-1} \right]_0^1 dy = \int_1^2 -\frac{1}{2} \left[\frac{1}{2x+3y} \right]_0^1 dy \\
 &= \int_1^2 -\frac{1}{2} \left[\frac{1}{2+3y} - \left(\frac{1}{3y} \right) \right] dy = -\frac{1}{2} \left[\int_1^2 \frac{1}{3} \left(\frac{3}{2+3y} \right) dy - \int_1^2 \frac{1}{3} \frac{3}{3y} dy \right] \\
 &= -\frac{1}{6} \left[\ln |2+3y| \Big|_1^2 - \ln |3y| \Big|_1^2 \right] = -\frac{1}{6} [\ln 8 - \ln 5 - \ln 6 + \ln 3] \\
 &= -\frac{1}{6} [\ln(4 \times 2) - \ln 5 - \ln(3 \times 2) + \ln 3] = -\frac{1}{6} [\ln 4 + \ln 2 - \ln 5 - \ln 3 - \ln 2 + \ln 3] \\
 &= -\frac{1}{6} [\ln 4 - \ln 5]
 \end{aligned}$$