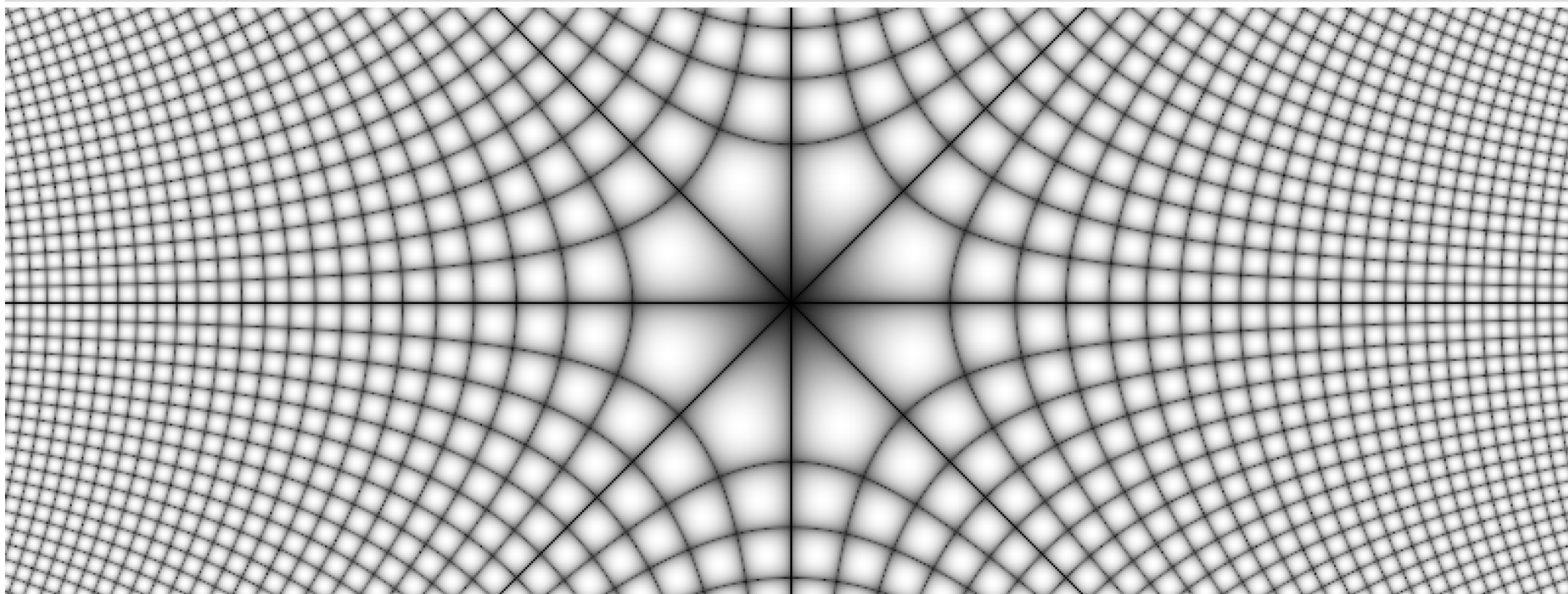


2.1: Complex functions



Complex functions

Let S be a set of complex numbers. A *function* f defined on S is a rule that assigns to each z in S a complex number w . The number w is called the value of f at z and is denoted by $f(z)$; that is, $w = f(z)$. The set S is called the domain of definition of f .

If only one value of w corresponds to each value of z , we say that w is a *single-valued* function of z or that $f(z)$ is single-valued. If more than one value of w corresponds to each value of z , we say that w is a *multiple-valued* or *many-valued* function of z .

A *multiple-valued* function can be considered as a collection of single-valued functions, each member of which is called a *branch* of the function. In general, we consider one particular member as a *principal branch* of the multiple-valued function and the value of the function corresponding to this branch as the *principal value*.

Example 2.1.1

The function $w = z^2$ is a single-valued function of z . On the other hand, if $w = z^{\frac{1}{2}}$, then to each value of z there are two values of w . Hence, the function

$$w = z^{\frac{1}{2}}$$

is a multiple-valued (in this case two-valued) function of z .

Suppose that $w = u + iv$ is the value of a function f at $z = x + iy$, so that

$$u + iv = f(x + iy)$$

Each of the real numbers u and v depends on the real variables x and y , and it follows that $f(z)$ can be expressed in terms of a pair of real-valued functions of the real variables x and y :

$$f(z) = u(x, y) + iv(x, y). \quad (2.1.1)$$

If the polar coordinates r and θ , instead of x and y , are used, then

$$u + iv = f(re^{i\theta})$$

where $w = u + iv$ and $z = re^{i\theta}$. In this case, we write

$$f(z) = u(r, \theta) + iv(r, \theta). \quad (2.1.2)$$

Example 2.1.2

Example 2: If $f(z) = z^2$ then

$$f(x + iy) = (x + iy)^2 = x^2 - y^2 + i(2xy).$$

Hence

$$u(x, y) = x^2 - y^2 \quad \text{and} \quad v(x, y) = 2xy.$$

When we use polar coordinates, we have

$$u(r, \theta) = r^2 \cos 2\theta \quad \text{and} \quad v(r, \theta) = r^2 \sin 2\theta.$$

Question: What happens when in either of equations (1) and (2) the function v always has a value zero?

Examples of complex functions

Polynomial functions

For $a_n, a_{n-1}, \dots, a_0$ complex constants we define

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$$

where $a_n \neq 0$ and n is a positive integer called the *degree* of the polynomial $p(z)$.

Rational functions: Ratios

$$\frac{p(z)}{q(z)}$$

where $p(z)$ and $q(z)$ are polynomials and $q(z) \neq 0$.

Exponential function

Exponential function: If $z = x + iy$, the exponential function e^z is defined by writing

$$e^z = e^x e^{iy}.$$

Because

$$e^{iy} = \cos y + i \sin y,$$

then we have

$$e^z = e^x (\cos y + i \sin y).$$

Logarithmic function

In a similar fashion, the complex logarithm is a complex extension of the usual real natural (i.e., base e) logarithm. In terms of polar coordinates $z = re^{i\theta}$, the complex logarithm has the form

$$\log z = \log(re^{i\theta}) = \log r + \log e^{i\theta} = \log r + i\theta.$$

We will explore in detail this function in the following section.

Trigonometric functions

The sine and cosine of a complex variable z are defined as follows:

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \quad \text{and} \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}.$$

The other four trigonometric functions are defined in terms of the sine and cosine functions with the following relations:

$$\begin{aligned} \tan z &= \frac{\sin z}{\cos z} & \cot z &= \frac{\cos z}{\sin z} \\ \sec z &= \frac{1}{\cos z} & \csc z &= \frac{1}{\sin z}. \end{aligned}$$

Hyperbolic trigonometric functions

The hyperbolic sine and the hyperbolic cosine of a complex variable are defined as they are with a real variable; that is,

$$\sinh z = \frac{e^z - e^{-z}}{2} \quad \text{and} \quad \cosh z = \frac{e^z + e^{-z}}{2}.$$

The other four hyperbolic functions are defined in terms of the hyperbolic sine and cosine functions with the relations:

$$\begin{aligned} \tanh z &= \frac{\sinh z}{\cosh z} & \coth z &= \frac{\cosh z}{\sinh z} \\ \operatorname{sech} z &= \frac{1}{\cosh z} & \operatorname{csch} z &= \frac{1}{\sinh z}. \end{aligned}$$

Explore the real and imaginary components

Use the following applet to explore the real and imaginary components of some complex functions.

INTERACTIVE GRAPH

Code

Enter the following scripts in [GeoGebra](#) to explore it yourself. Open the 3D view. The symbol `#` indicates comments.

```
#Define complex function

f(z) := z + 1/z

#Define components

Re = Surface(u, v, real( f(u + i v) ), u, -5, 5, v, -5, 5)

Im = Surface(u, v, imaginary( f(u + i v) ), u, -5, 5, v, -5, 5)
```

This page titled [2.1: Complex functions](#) is shared under a [CC BY-NC-SA](#) license and was authored, remixed, and/or curated by [Juan Carlos Ponce Campuzano](#).