

Functions of a Complex Variable

Traditionally, in mathematics, complex analysis is known as the theory of functions of a complex variable. It is the branch of mathematical analysis that analyses functions of complex numbers. It is helpful in multiple branches of mathematics, including number theory, algebraic geometry, analytic combinatorics, and applied mathematics. In this article, you will learn what the functions of a complex variable are, and how to solve the functions of a complex variable.

Definition of Functions of a Complex Variable

Complex variable functions or complex functions are functions that assign complex numbers for complex numbers. Let C be the [set](#) of complex numbers.

A function $f : C \rightarrow C$ is a rule which associates with $z \in C$, a unique $w \in C$, written as $w = f(z)$.

Here,

$$z = x + iy$$

$$w = u + iv, \text{ where } u = u(x, y) \text{ and } v = v(x, y)$$

Thus, u and v are functions of x and y .

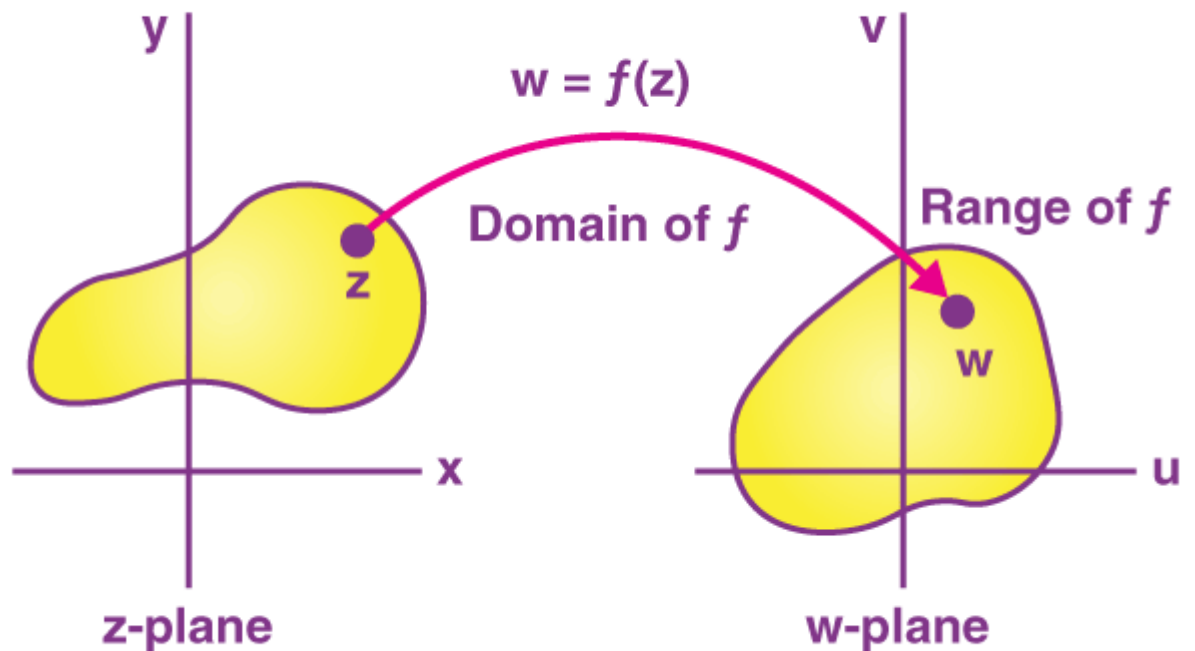
$$w = f(z) = u(x, y) + i v(x, y)$$

From this we can say that the standard notations for real and imaginary parts of z are x and y . Also, the standard notations for real and imaginary parts of w are u and v , where u and v are functions of x and y .

Note:

A complex valued function does not have to be defined on the whole of C ; it may be defined over a non-empty subset D of C . Here, D is called the domain of definition of the function f .

A function of a complex variable, i.e., $w = f(z)$ is generally expressed as a mapping or transformation from the complex plane z to the complex plane w . This can be shown as:



The graph of the function often reveals properties of a real-valued function of a real variable. However, no suitable graphical representation is available for a function, $w = f(z)$, where z and w are complex since the numbers z and w are located in a plane rather than a line.

Solved Examples

Example 1:

Consider $f(z) = z^2 + iz$, and express it in terms of real and imaginary parts?

Solution:

We know that $z = x + iy$

Now, $f(z) = z^2 + iz$

$$= (x + iy)^2 + i(x + iy)$$

$$= x^2 + i^2y^2 + 2ixy + ix + i^2y$$

$$= x^2 - y^2 + 2ixy + ix - y \text{ \{since } i^2 = -1\}}$$

$$= x^2 - y^2 - y + i(x + 2xy)$$

This is of the form $w = u + iv$ such that $u = x^2 - y^2 - y$ and $v = x + 2xy$.

Or

$$u = u(x, y) = x^2 - y^2 - y$$

$$v = v(x, y) = x + 2xy$$

Example 2:

Find $u(x, y)$ and $v(x, y)$ for the complex function $f(z) = z^3$?

Solution:

Given, $f(z) = z^3$

As we know, $z = x + iy$

$$\text{So, } f(z) = (x + iy)^3$$

$$= x^3 + i^3y^3 + 3ixy(x + iy)$$

$$= x^3 - iy^3 + 3ix^2y + 3i^2xy^2 \text{ \{since } i^3 = -i\}}$$

$$= x^3 - iy^3 + 3ix^2y - 3xy^2 \text{ \{since } i^2 = -1\}}$$

$$= (x^3 - 3xy^2) + i(3x^2y - y^3)$$

This is of the form $w = f(z) = u + iv$.

$$\text{So, } u = u(x, y) = x^3 - 3xy^2$$

$$v = v(x, y) = 3x^2y - y^3$$

Example 3:

For what values of z is the complex function $w = 1/z$ defined? Also, find u , v and w when $z = 3 - 2i$?

Solution:

We know that, $z = x + iy$

Given, $w = 1/z$

Thus, w is defined for all values of $z \neq 0$.

Now,

$$w = 1/z$$

$$= 1/(x + iy)$$

By rationalizing the denominator, we get;

$$w = [1/(x + iy)] \cdot [(x - iy)/(x - iy)]$$

$$= (x - iy)/(x^2 - i^2y^2)$$

$$= (x - iy)/(x^2 + y^2) \text{ \{since } i^2 = -1\}}$$

This can be written as:

$$w = x/(x^2 + y^2) - iy/(x^2 + y^2)$$

Comparing with $w = u + iv$, we have

$$u = u(x, y) = x/(x^2 + y^2)$$

$$v = v(x, y) = -y/(x^2 + y^2)$$

Consider $z = 3 - 2i$

By comparing with $z = x + iy$, we get $x = 3$ and $y = -2$

$$\text{And } x^2 + y^2 = (3)^2 + (-2)^2 = 9 + 4 = 13$$

$$\text{So, } u = 3/13 \text{ and } v = -(-2)/13 = 2/13$$

$$\text{Therefore, } w = (3/13) + i(2/13)$$

Find the real and imaginary parts of $f(z) = z^2 + 3z$.

$$f(z) = z^2 + 3z$$

$$\Rightarrow f(z) = (x + iy)^2 + 3(x + iy)$$

$$\Rightarrow f(z) = x^2 + 2ixy - y^2 + 3x + 3iy$$

$$\Rightarrow f(z) = (x^2 - y^2 + 3x) + i(3y + 2xy)$$

$$\Rightarrow \text{Re}[f(z)] = (x^2 - y^2 + 3x) \text{ \& } \text{Im}[f(z)] = (3y + 2xy)$$

□

Determine & sketch the image $|z| = 1$ of under the transformation $w = z + i$.

$$\text{Let, } w = z + i$$

$$\Rightarrow u + iv = (x + iy) + i$$

$$\Rightarrow u + iv = x + i(y + 1)$$

$$\Rightarrow u = x \text{ \& } v = y + 1$$

$$\bullet |z| = 1$$

$$\Rightarrow \sqrt{x^2 + y^2} = 1 \Rightarrow x^2 + y^2 = 1$$

$$\Rightarrow u^2 + (v - 1)^2 = 1$$

Practice Problems

1. Find the real and imaginary parts of $w = f(z) = z^2 - z$?
2. Solve: $f(z) = \sin z$?
3. Find the functions of u and v for $w = az^2 + bz + c$?

Complex Numbers

Decompose $f(z) = z^2$ into its real and imaginary parts.

Solution:

$$\text{Given: } w = f(z) = z^2$$

Let $z = x + iy$, as per the standard form of a complex number.

$$\begin{aligned}\Rightarrow z^2 &= (x + iy)^2 = x^2 + 2(x)(iy) + (iy)^2 \\ &= x^2 - y^2 + i(2xy)\end{aligned}$$

Hence, the real part is $x^2 - y^2$ and the imaginary part is $2xy$.

Similar Problems:

Question 1: Find the real and imaginary parts of $(1 + i)^2$.

Solution:

$$(1 + i)^2 = 1^2 + i^2 + 2i$$

$$= 1 - 1 + 2i$$

$$= 2i$$

Thus, $(1 + i)^2$ has no real part and the imaginary part is 2.

Question 2: Find the real and imaginary parts of $(1 - i)^2$.

Solution:

$$(1 - i)^2 = 1^2 + i^2 - 2i$$

$$= 1 - 1 - 2i$$

$$= -2i$$

The real part of $(1 - i)^2$ is 0 and the imaginary part is -2i.

Question 3: Find the real and imaginary parts of $(3 + 2i)^2$.

Solution:

$$(3 + 2i)^2 = 3^2 + 2(2i)(3) + (2i)^2$$

$$= 9 + 4(-1) + 12i$$

$$= 5 + 12i$$

Thus, real part is 5 and the imaginary part is 12.

Question 4: Find the real and imaginary parts of $(12 + 6i)^2$.

Solution:

$$(12 + 6i)^2 = 12^2 + 2(12)(6i) + (6i)^2$$

$$= 144 + 36(-1) + 144i$$

$$= 108 + 144i$$

Thus, real part is 108 and the imaginary part is 144.

Question 5: Find the real and imaginary parts of $(10 + 10i)^2$.

Solution:

$$(10 + 10i)^2 = 10^2 + 2(10)(10i) + (10i)^2$$

$$= 100 + 200(-1) + 100i$$

$$= -100 + 100i$$

Thus, real part is -100 and the imaginary part is 100.

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