

وزارة التعليم العالي والبحث العلمي

كلية المأمون الجامعة

قسم هندسة تقنيات الحاسبات

رياضيات

الملزمة -2-

المرحلة الاولى

مدرس المادة

أستاذة نبراس

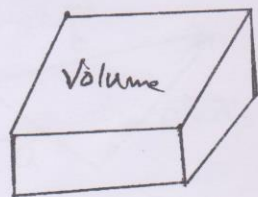
مكتبة عادل

Vectors and Scalars

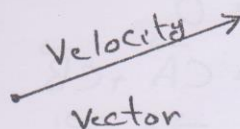
In physics and geometry and its engineering applications we use two kinds of quantities, scalars and vectors where

Scalars are quantities that are fully described by a magnitude (or numerical value) alone. For instance length, area, volume, temperature, work --- are scalars

Vectors are quantities that are fully described by both a magnitude and a direction where displacement, velocity, acceleration, momentum, force --- are vectors



Scalar



We denote vectors by $A, B, C, \dots$ or you may use arrows for instance $\vec{A}$ (in place of A), $\vec{B}$ (in place of B)

(1)

A vector (arrow) has an initial point (P) and terminal point (Q)



Definition The length (magnitude) of a vector $\vec{PQ}$ is equal to the distance between P and Q and denoted by $|PQ|$

If $\vec{A}, \vec{B}, \vec{W}$ are vectors, C, K are scalars then,

$$\vec{A} + \vec{B} = \vec{B} + \vec{A} \quad [\text{commutativity}]$$

$$(\vec{A} + \vec{B}) + \vec{W} = \vec{A} + (\vec{B} + \vec{W}) \quad [\text{Associativity}]$$

$$\vec{A} + \vec{0} = \vec{A} = \vec{0} + \vec{A}$$

$$\vec{A} + (-\vec{A}) = \vec{0}$$

$$C(\vec{A} + \vec{B}) = C\vec{A} + C\vec{B}$$

$$(C+K)\vec{A} = C\vec{A} + K\vec{A}$$

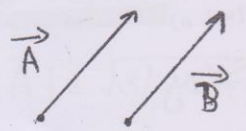
$$C(K\vec{A}) = (CK)\vec{A}$$

Def: Unit vector: A vector of length "1" and denoted by $\hat{A} = \frac{\vec{A}}{|\vec{A}|}$

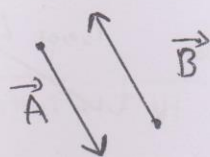
(2)

Equality of Vectors

Two vectors A and B are equal written $\vec{A} = \vec{B}$, if they have the same length and the same direction



equal vector

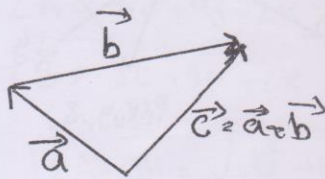


$$\vec{A} = -\vec{B}$$

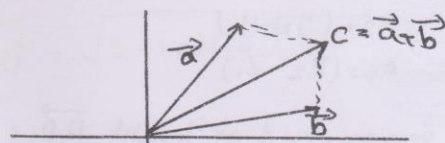
vectors having the same length but different direction

Vector Addition, Scalar Multiplication

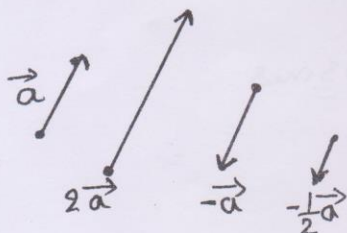
The sum of two vectors a, b is obtained by adding the corresponding components



Vector addition



(Parallelogram law)

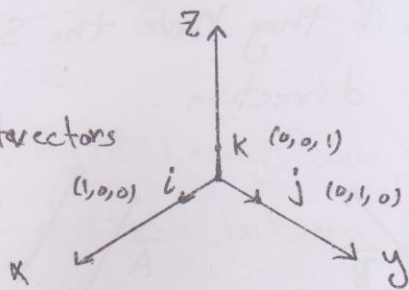


(3)

Components of a vector

We choose an xyz Cartesian coordinate where

$\hat{i}, \hat{j}, \hat{k}$ are the unit vectors in the positive direction of the axes of Cartesian



$$\text{Let } \vec{OP_1} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$$

$$\vec{OP_2} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$$

$$\Rightarrow \vec{P_1P_2} = \vec{OP_2} - \vec{OP_1}$$

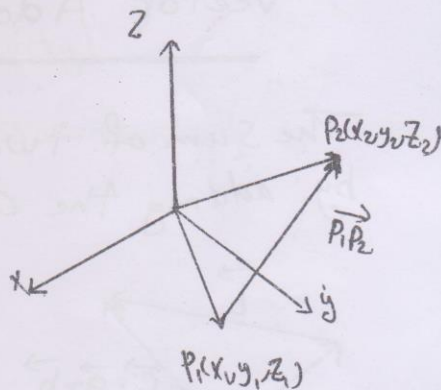
$$= (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$

$$\vec{r} = \vec{P_1P_2} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

$$\text{where } a_1 = (x_2 - x_1)$$

$$a_2 = (y_2 - y_1)$$

$$a_3 = (z_2 - z_1)$$



$$\Rightarrow \text{The magnitude or length } |\vec{P_1P_2}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

and the direction cosine of the position vector

$$\vec{r} = |\vec{r}| \cos \alpha \hat{i} + |\vec{r}| \cos \beta \hat{j} + |\vec{r}| \cos \gamma \hat{k}$$

where (α, β, γ) direction cosines

$$|\hat{r}| = \sqrt{\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma} = 1$$

(4)

Example If $\vec{A} = 3\hat{i} + 2\hat{j} - \hat{k}$, $\vec{B} = \hat{i} + \hat{j} + 2\hat{k}$
 Find the magnitude of $A, B, A+B, A-B$
 and $(2A-3B)$, then find U_{A-B}

Solution:

$$|\vec{A}| = \sqrt{3^2 + 2^2 + (-1)^2} = \sqrt{9+4+1} = \sqrt{14}$$

$$|\vec{B}| = \sqrt{1^2 + 1^2 + 2^2} = \sqrt{1+1+4} = \sqrt{6}$$

$$\vec{A} + \vec{B} = 4\hat{i} + 3\hat{j} + \hat{k}$$

$$|A+B| = \sqrt{4^2 + 3^2 + 1^2} = \sqrt{16+9+1} = \sqrt{26}$$

$$\vec{A} - \vec{B} = 2\hat{i} + \hat{j} - 3\hat{k}$$

$$|A-B| = \sqrt{2^2 + 1^2 + (-3)^2} = \sqrt{4+1+9} = \sqrt{14}$$

$$\begin{array}{l} 2\vec{A} = 6\hat{i} + 4\hat{j} - 2\hat{k} \\ 3\vec{B} = 3\hat{i} + 3\hat{j} + 6\hat{k} \end{array} \rightarrow 2\vec{A} - 3\vec{B} = 3\hat{i} - \hat{j} - 8\hat{k}$$

$$\therefore |2A-3B| = \sqrt{3^2 + (-1)^2 + (-8)^2} = \sqrt{9+1+64} = \sqrt{74}$$

$$U_{AB} = \frac{\vec{A} \cdot \vec{B}}{|\vec{A}| |\vec{B}|} = \frac{2\hat{i} + \hat{j} - 3\hat{k}}{\sqrt{14}}$$

$$= \frac{2}{\sqrt{14}} \hat{i} + \frac{1}{\sqrt{14}} \hat{j} - \frac{3}{\sqrt{14}} \hat{k}$$

(5)

Example: If $\vec{A} = 4\hat{i} + 2\hat{k}$, $\vec{B} = 6\hat{i} - \hat{j} + 2\hat{k}$

then find $|\vec{AB}|$, $\mathcal{U}_{\vec{AB}}$

Solution

First we have to find $\vec{AB}$

$$\vec{AB} = (6-4)\hat{i} + (-1-0)\hat{j} + (2-2)\hat{k}$$

$$\vec{AB} = 2\hat{i} - \hat{j}$$

$$\therefore |\vec{AB}| = \sqrt{(2)^2 + (-1)^2} = \sqrt{5}$$

$$\text{and } \mathcal{U}_{\vec{AB}} = \hat{AB} = \frac{\vec{AB}}{|\vec{AB}|} = \frac{2\hat{i} - \hat{j}}{\sqrt{5}}$$

$$\hat{AB} = \frac{2}{\sqrt{5}}\hat{i} - \frac{1}{\sqrt{5}}\hat{j}$$

Ex For vectors $\vec{a} = 3\hat{i} + 4\hat{j}$ and $\vec{b} = 2\hat{i} - \hat{j}$ find the magnitude and direction of (a) $\vec{a} \otimes \vec{b}$ (b) $\vec{b} - \vec{a}$

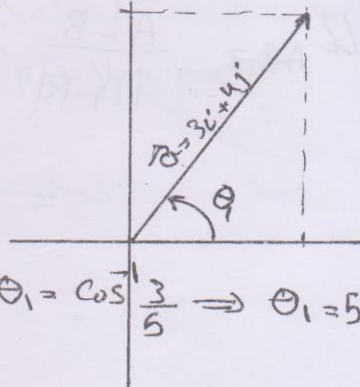
Solution

$$|\vec{a}| = \sqrt{(3)^2 + (4)^2} = \sqrt{9+16} = 5$$

the direction of $\vec{a}$ is

$$\alpha_1 = |\alpha| \cos \theta_1 \Rightarrow \cos \theta_1 = \frac{3}{5} \Rightarrow \theta_1 = \cos^{-1} \frac{3}{5} \Rightarrow \theta_1 = 53^\circ$$

(6)

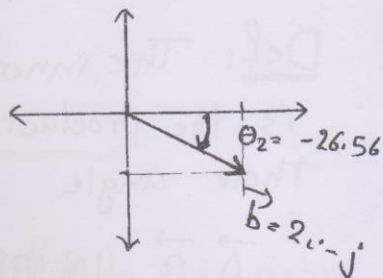


The magnitude of $\vec{b}$; $|\vec{b}| = \sqrt{2^2 + (-1)^2} = \sqrt{5}$

and $\alpha_1 = |\vec{b}| \cos \theta_2$

$$2 = \sqrt{5} \cos \theta_2$$

$$\Rightarrow \cos \theta_2 = \frac{2}{\sqrt{5}} \Rightarrow \theta_2 = \cos^{-1} \frac{2}{\sqrt{5}}$$



$$\therefore \theta_2 = -26.56 \Rightarrow \theta_2 = 360 - 26.56 = 333.44$$

the mag. of $|\vec{b} - \vec{a}|$ is

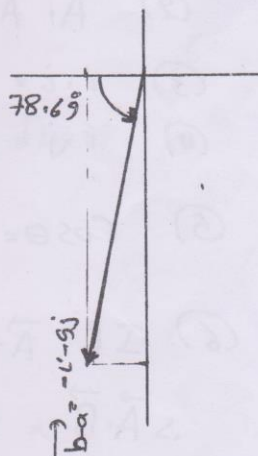
$$\vec{b} - \vec{a} = (2\vec{i} - \vec{j}) - (3\vec{i} + 4\vec{j})$$

$$\vec{b} - \vec{a} = -\vec{i} - 5\vec{j} \Rightarrow |\vec{b} - \vec{a}| = \sqrt{(-1)^2 + (-5)^2} = \sqrt{26}$$

$$\cos \theta_3 = \frac{-1}{\sqrt{26}}$$

$$\theta_3 = \cos^{-1} \frac{-1}{\sqrt{26}} \Rightarrow \theta_3 = 78.69$$

$$\therefore \theta_3 = 180 + 78.69 = 258.69$$



(7)

Inner Product (Dot Product)

Def: The inner product of two vectors A and B is the product of their lengths times cosine of their angle

$$\vec{A} \cdot \vec{B} = |\vec{A}| |\vec{B}| \cos \theta \quad \vec{A} \neq 0, \vec{B} \neq 0$$

$$\vec{A} \cdot \vec{B} = 0 \quad \text{If } \vec{A} = 0 \text{ or } \vec{B} = 0 \text{ or } \vec{A} \perp \vec{B}$$

Notes we have

$$(1) \vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$$

$$(2) \vec{A} \cdot \vec{A} = |\vec{A}| |\vec{A}| = |\vec{A}|^2$$

$$(3) \vec{i} \cdot \vec{i} = \vec{j} \cdot \vec{j} = \vec{k} \cdot \vec{k} = 1 \quad \text{Since } \cos 0 = 1$$

$$(4) \vec{i} \cdot \vec{j} = \vec{j} \cdot \vec{k} = \vec{k} \cdot \vec{i} = 0 \quad \text{Since } \cos 90 = 0$$

$$(5) \cos \theta = \frac{\vec{A} \cdot \vec{B}}{|\vec{A}| |\vec{B}|}$$

$$(6) \text{ If } \vec{A} = a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}, \vec{B} = b_1 \vec{i} + b_2 \vec{j} + b_3 \vec{k} \\ \Rightarrow \vec{A} \cdot \vec{B} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

$$(7) \text{ The projection of } \vec{B} \text{ on } \vec{A} = |\vec{B}| \cos \theta \\ = |\vec{B}| \frac{\vec{A} \cdot \vec{B}}{|\vec{A}| |\vec{B}|} = \frac{\vec{A} \cdot \vec{B}}{|\vec{A}|}$$

(8)

and the projection of $\vec{A}$ on $\vec{B} = |\vec{A}| \cos \theta$

$$= |\vec{A}| \frac{\vec{A} \cdot \vec{B}}{|\vec{A}| |\vec{B}|} = \frac{\vec{A} \cdot \vec{B}}{|\vec{B}|}$$

Example Find $\vec{A} \cdot \vec{B}$ and find the angle between the vectors $\vec{A} = i + 2j$, $\vec{B} = 3i - 2j + k$

Solution

$$\vec{A} \cdot \vec{B} = |\vec{A}| |\vec{B}| \cos \theta$$

$$\begin{aligned} \vec{A} \cdot \vec{B} &= a_1 b_1 + a_2 b_2 + a_3 b_3 \\ &= (1)(3) + (2)(-2) + (0)(1) \\ &= 3 - 4 = -1 \end{aligned}$$

$$|\vec{A}| = \sqrt{1+4} = \sqrt{5}, \quad |\vec{B}| = \sqrt{9+4+1} = \sqrt{14}$$

$$\cos \theta = \frac{\vec{A} \cdot \vec{B}}{|\vec{A}| |\vec{B}|} = \frac{-1}{\sqrt{5} \sqrt{14}} = \frac{-1}{\sqrt{70}}$$

$$\theta = \cos^{-1} \frac{-1}{\sqrt{70}} \Rightarrow \theta = \cos^{-1}(-0.11925)$$

$$\theta = 96.865^\circ$$

(9)

Example Find the angle between $\vec{A} = i - 2j - 2k$
and $\vec{B} = 6i + 3j + 2k$

Solution $\cos \theta = \frac{\vec{A} \cdot \vec{B}}{|\vec{A}| |\vec{B}|}$

$$\begin{aligned}\vec{A} \cdot \vec{B} &= a_1 b_1 + a_2 b_2 + a_3 b_3 \\ &= (1)(6) + (-2)(3) + (-2)(2) \\ &= 6 - 6 - 4 = -4\end{aligned}$$

$$|\vec{A}| = \sqrt{1+4+4} = \sqrt{9} = 3, \quad |\vec{B}| = \sqrt{36+9+4} = \sqrt{49} = 7$$

$$\cos \theta = \frac{-4}{(3)(7)} \Rightarrow \theta = \cos^{-1} \frac{-4}{21} \Rightarrow \theta = 100.98^\circ$$

Ex Find the projection of A on B or
(Component of A in the direction of B) and
the projection of B on A, $\vec{A} = 5j + 2k$ / $\vec{B} = 4j - 3k$

Solution

$$\text{Proj } \vec{A} \text{ on } \vec{B} = \frac{\vec{A} \cdot \vec{B}}{|\vec{B}|} = \frac{(5)(4) + (2)(-3)}{\sqrt{4^2 + (-3)^2}} = \frac{20 - 6}{\sqrt{16+9}} = \frac{14}{5}$$

$$\text{Proj } \vec{B} \text{ on } \vec{A} = \frac{\vec{A} \cdot \vec{B}}{|\vec{A}|} = \frac{14}{\sqrt{5^2 + 2^2}} = \frac{14}{\sqrt{25+4}} = \frac{14}{\sqrt{29}}$$

(10)

Vector Product (Cross product)

Def: The vector product (cross product) $\vec{A} \times \vec{B}$ of two vectors A and B is defined by

$$\vec{A} \times \vec{B} = |\vec{A}| |\vec{B}| \sin \theta \hat{n}$$

θ is the smallest angle between A and B and $\hat{n}$ is a vector perpendicular on both $\vec{A}$ and $\vec{B}$

So,

$|\vec{A} \times \vec{B}| = |\vec{A}| |\vec{B}| \sin \theta = \text{The area of the Parallelogram}$

notes

(1) $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$

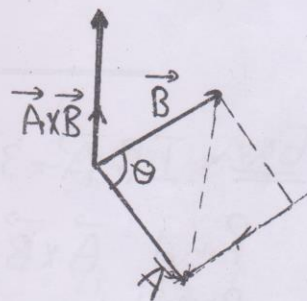
(2) $\vec{A} \times \vec{B} = \text{vector}$
 $\vec{A} \cdot \vec{B} = \text{Scalar}$

(3) If $\vec{A} \times \vec{B} = 0$ $\vec{A} = 0$, or $\vec{B} = 0$ or $\vec{A} \parallel \vec{B}$

(4) $i \times j = k$, $j \times k = i$, $k \times i = j$

(5) (a) $\vec{A} \times (\vec{B} + \vec{C}) = (\vec{A} \times \vec{B}) + (\vec{A} \times \vec{C})$

(b) $(\vec{A} + \vec{B}) \times \vec{C} = (\vec{A} \times \vec{C}) + (\vec{B} \times \vec{C})$
(11)



$$(c) \vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \times \vec{B}) \times \vec{C}$$

$$(d) \vec{A} \cdot (\vec{B} \times \vec{C}) = -\vec{B} \cdot (\vec{A} \times \vec{C})$$

$$(6) \text{ If } \vec{A} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}, \vec{B} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$$

then

$$\vec{A} \times \vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

(7)

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = (\vec{A} \times \vec{B}) \cdot \vec{C}$$

Ex If $\vec{A} = 3\vec{i} + 2\vec{j} - \vec{k}$, $\vec{B} = 5\vec{i} - 2\vec{j} + \vec{k}$

Find $\vec{A} \times \vec{B}$ and find the area of the Parallelogram

Solu:

$$\vec{A} \times \vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 2 & -1 \\ 5 & -2 & 1 \end{vmatrix} = \vec{i} \begin{vmatrix} 2 & -1 \\ -2 & 1 \end{vmatrix} - \vec{j} \begin{vmatrix} 3 & -1 \\ 5 & 1 \end{vmatrix} + \vec{k} \begin{vmatrix} 3 & 2 \\ 5 & -2 \end{vmatrix}$$

$$= 0\vec{i} - 8\vec{j} - 16\vec{k} \Rightarrow \vec{A} \times \vec{B} = -8\vec{j} - 16\vec{k}$$

(12)

$$\text{and } \vec{B} \times \vec{A} = -(\vec{A} \times \vec{B}) = -(-8\vec{j} - 16\vec{k}) = 8\vec{j} + 16\vec{k}$$

$$\text{then } |\vec{A} \times \vec{B}| = \sqrt{8^2 + 16^2} = \sqrt{64 + 256} = \sqrt{320} \approx 17.888$$

Square Unit

Ex Find the Parallelogram of the vectors

$$\vec{A} = 4\vec{i} - \vec{j} - \vec{k}, \quad \vec{B} = \vec{i} + 2\vec{j}$$

Solu,

$$\vec{A} \times \vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & -1 & -1 \\ 1 & 2 & 0 \end{vmatrix} = \vec{i} \begin{vmatrix} -1 & -1 \\ 2 & 0 \end{vmatrix} - \vec{j} \begin{vmatrix} 4 & -1 \\ 1 & 0 \end{vmatrix} + \vec{k} \begin{vmatrix} 4 & -1 \\ 1 & 2 \end{vmatrix}$$

$$\vec{A} \times \vec{B} = 2\vec{i} - \vec{j} + 9\vec{k}$$

$$\text{The area} = |\vec{A} \times \vec{B}| = \sqrt{4 + 1 + 81} = \sqrt{86} \text{ Square units}$$

Ex If $\vec{A} = 4\vec{i} - \vec{j} - \vec{k}$, $\vec{B} = \vec{i} + 2\vec{j}$, $\vec{C} = 2\vec{i} - 3\vec{k}$

Find $\vec{A} \cdot (\vec{B} \times \vec{C})$

Solution

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \begin{vmatrix} 4 & -1 & -1 \\ 1 & 2 & 0 \\ 2 & 0 & -3 \end{vmatrix}$$

$$= 4 \begin{vmatrix} 2 & 0 \\ 0 & -3 \end{vmatrix} - 3 \begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix} = 4(-6) - 3(-4) = -24 + 12 = -12$$

(13)

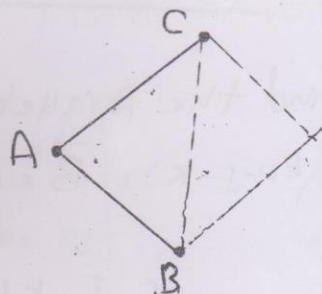
Ex Find the area of the Triangle whose vertices are $A(1, -1, 0)$, $B(2, 1, -1)$, $C(-1, 1, 2)$

Solution

$$\vec{AB} = (2-1)\mathbf{i} + (1+1)\mathbf{j} + (-1-0)\mathbf{k}$$

$$\vec{AB} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$$

$$\vec{AC} = (-1-1)\mathbf{i} + (1+1)\mathbf{j} + (2-0)\mathbf{k}$$
$$= -2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$$



The area of Triangle $= \frac{1}{2} |\vec{AB} \times \vec{AC}|$
 $= \frac{1}{2}$ area of Parallelogram

$$|\vec{AB} \times \vec{AC}| = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ -2 & 2 & 2 \end{vmatrix} = 6\mathbf{i} + 0\mathbf{j} + 6\mathbf{k}$$

$$|\vec{AB} \times \vec{AC}| = \sqrt{36+36} = \sqrt{72} = \sqrt{9 \times 8} = 6\sqrt{2}$$

$$\therefore \frac{1}{2} |\vec{AB} \times \vec{AC}| = \frac{1}{2} 6\sqrt{2} = 3\sqrt{2}$$

(14)

" Matrices and Determinants "

Q₁ Find the elements C_{33} , C_{32} in product,

$$AB = C \text{ where } A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ -1 & 1 & 2 \end{bmatrix}; B = \begin{bmatrix} 1 & 3 & 0 \\ -1 & 2 & -1 \\ 0 & 0 & 2 \end{bmatrix}$$

Q₂ let $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}; B = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$

a) Find $A \cdot B$ b) prove that $A^3 = 4A$

Q₃ let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$ Find $\text{adj}(A)$

Q₄ Find $\begin{bmatrix} 2 & 3 & 1 \\ -1 & 2 & 4 \end{bmatrix} \times \begin{bmatrix} 5 & 3 \\ 4 & 0 \\ 3 & 1 \end{bmatrix}$

Q₅ Evaluate the following product;
 $\begin{bmatrix} 3 & 6 & 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}; \begin{bmatrix} 2 \\ 3 \end{bmatrix} \times \begin{bmatrix} 4 & 5 \end{bmatrix}$

Q₆ let $A = \begin{bmatrix} 1 & 2 & 4 \\ -1 & 0 & 3 \\ 3 & 1 & -2 \end{bmatrix}$ Find A^{-1}

Q₇ let $A = \begin{bmatrix} 2 & 0 & 1 \\ 3 & -1 & 4 \\ 5 & -2 & 1 \end{bmatrix}$ prove that $AA^{-1} = I$

Q₈ IF $A = \begin{bmatrix} 3 & -1 & 2 \\ 4 & -1 & 2 \\ 3 & 1 & -2 \end{bmatrix}$ prove that $A(A^{-1}) = I$

Q9 Find the inverse of A (A^{-1}) where

1- $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$ 2- $\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$

Q10 let $A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 2 \\ 3 & 1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$; $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Find (a) $|A|$ (b) $\text{cof } A$ (c) A^{-1}

(d) values of x_1, x_2, x_3 where $X = A^{-1}B$

Q11 Solve by the Grammer's rules

(a)
$$\begin{aligned} -2x_1 + 3x_2 - x_3 &= 1 \\ x_1 + 2x_2 - x_3 &= 4 \\ -2x_1 + x_3 - x_2 &= 3 \end{aligned}$$

(b)
$$\begin{aligned} 2x_1 &= -2 + 5x_2 \\ 6x_2 + 4x_1 &= 1 \end{aligned}$$

(c)
$$\begin{aligned} x + 2y + z &= 3 \\ 2x + y &= z \\ x + z &= y \end{aligned}$$