

وزارة التعليم العالي والبحث العلمي

كلية المأمون الجامعة

قسم هندسة تقنيات الحاسبات

رياضيات

الملزمة [1]

المرحلة الاولى

مكتبة عادل

Chapter (1) Matrices and Determinants

Matrices

2.1 Definition :

A rectangular array of real or complex numbers of the form:

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ a_{31} & a_{32} & \dots & a_{3n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix} \dots (1)$$

is called a matrix (Fig. 1.1)

- The numbers $a_{11}, \dots, a_{mn}$ are called the elements of the matrix.
- The horizontal lines are called rows.
- The vertical lines are called columns.
- A matrix with (m) rows and (n) columns is called an $(m \times n)$ matrix (read m by n matrix).
- Matrices will be denoted by capital bold-faced letters $A, B, \dots$ etc, or by $[a_{jk}], [b_{jk}], \dots$ etc.
- A matrix $[a_1 \ a_2 \ \dots \ a_n]$ having only one row is called a Row Matrix or Row Vector.
- A vector having only one column is called a column matrix or Column Vector

$$\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

(1)

- The elements of a matrix may be real or complex numbers. If all the elements of a matrix are real the matrix is called a Real Matrix.

- A matrix having the same number of rows and columns is called (a Square matrix), and the number of rows is called its order. A square matrix of order (n) is called an $(n \times n)$ matrix (read n by n matrix).

2.2 Types of Matrices:

2.2.1 Square Matrix:

number of rows = number of columns

$$\begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix}; \begin{bmatrix} -1 & 3 & 2 \\ 0 & 7 & 1 \\ 2 & 4 & 5 \end{bmatrix}$$

2.2.2 Diagonal Matrix:

It is Square Matrix, all the elements are zero except the elements of the main diagonal are not zero.

$$\begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}; \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

2.2.3 Triangular Matrix:

It is a square Matrix whose elements upper or lower

If $i > j \Rightarrow a_{ij} = 0$ (upper)

OR $i < j \Rightarrow a_{ij} = 0$ (Lower)

$$\begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}; \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}; \begin{bmatrix} 1 & 0 & 0 \\ 2 & 5 & 0 \\ 4 & 2 & 1 \end{bmatrix}; \begin{bmatrix} 1 & 7 & 2 \\ 0 & 3 & - \\ 0 & 0 & 2 \end{bmatrix}$$

2.2.4 Identity Matrix

It is a diagonal Matrix whose elements on the main diagonal are all (1) and denoted by (I_n) .

$$I_1 = [1]; I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2.2.5 Zero Matrix:

All the elements are Zero

$$[0]; \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}; \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

2.3 Mathematical operations of Matrices

let:

$$A = [a_{jk}]_{m \times n} \text{ and } B = [b_{jk}]_{m \times n} \dots (2)$$

(3)

2.3.1 Equality:

Two matrices A and B are equal if and only if A and B have the same numbers of rows and the same number of columns and corresponding elements are equal that is

$$a_{jk} = b_{jk} \text{ for all occurring } j \text{ and } k.$$

2.3.2 Addition of Matrices:

Is defined only for matrices having the same number of rows and columns and is defined as follows:

The sum of two $(m \times n)$ matrices $A = [a_{jk}]$ and $B = [b_{jk}]$ is the $(m \times n)$ matrix $C = [c_{jk}]$ with elements;

$$c_{jk} = a_{jk} + b_{jk} \quad j = 1, 2, \dots, m \quad (3)$$

$$k = 1, 2, \dots, n$$

and written $C = A + B$

EX. 1 Addition of Matrices

$$A = \begin{bmatrix} -4 & 6 & 3 \\ 0 & 1 & 2 \end{bmatrix}; B = \begin{bmatrix} 5 & -1 & 0 \\ 3 & 1 & 0 \end{bmatrix}$$

Then

$$A + B = \begin{bmatrix} 1 & 5 & 3 \\ 3 & 2 & 2 \end{bmatrix}$$

(4)

3.2.1 properties of Matrix addition

$$\left. \begin{aligned} A+B &= B+A \\ A+O &= A \\ A+(-A) &= O \end{aligned} \right\} \text{--- (4)}$$

where $-A = [-a_{jk}]$ is the $(m \times n)$ matrix obtained by multiplying every element of A by (-1) and is called the negative of A .

2.3.3 Multiplication of Matrices by Scalars (number)

The product of an $(m \times n)$ matrix (A) by a scalar q (a number) is denoted by (qA) or (Aq) and is the $(m \times n)$ matrix obtained by multiplying every element of (A) by (q) .

$$qA = Aq = \begin{bmatrix} qa_{11} & qa_{12} & \dots & qa_{1n} \\ qa_{21} & qa_{22} & \dots & qa_{2n} \\ qa_{m1} & qa_{m2} & \dots & qa_{mn} \end{bmatrix}$$

EX. 2

If $A = \begin{bmatrix} 2.7 & -1.8 \\ 0.9 & 3.6 \end{bmatrix}$ Calculate $2A$ & $\frac{10}{9}A$?

Solution:

$$2A = \begin{bmatrix} 5.4 & -3.6 \\ 1.8 & 7.2 \end{bmatrix}$$

and $\frac{10}{9}A = \begin{bmatrix} 3 & -2 \\ 1 & 4 \end{bmatrix}$

(5)

2.3.3.1 Properties of Matrix Multiplication by Scalars:

$$\left. \begin{aligned} \eta(A+B) &= \eta A + \eta B \\ (C+K) \cdot A &= CA + KA \\ C(KA) &= CK(A) \\ 1 \cdot A &= A \end{aligned} \right\} \text{-----(5)}$$

2.4 Transpose of A Matrix:

The matrix of order $(n \times m)$ obtained by interchanging the rows and columns of an $(m \times n)$ matrix (A) is called the transpose of (A) and is denoted (A^T) .

i.e.

$$\text{The transpose of } A = [a_{jk}] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

$$\text{is } A^T = [a_{jk}] = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{bmatrix} \text{---(6)}$$

For example:

$$\text{The transpose of } A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

(6)

A real square matrix $A = [a_{jk}]$ is said to be symmetric if it is equal to its transpose, $A^T = A$; that is $a_{kj} = a_{jk}$

- A real square matrix $A = [a_{jk}]$ is said to be skew-symmetric if $A^T = -A$, that is $a_{kj} = -a_{jk}$. for example;

$$A = \begin{bmatrix} 0 & -4 & 1 \\ 4 & 0 & -5 \\ -1 & 5 & 0 \end{bmatrix} \quad A^T = \begin{bmatrix} 0 & 4 & -1 \\ -4 & 0 & 5 \\ 1 & -5 & 0 \end{bmatrix}$$

$$\therefore \boxed{A^T = -A}$$

- A real square matrix (A) may be written as the ~~same~~ sum of a symmetric matrix (R) and a skew-symmetric (S), where

$$\left. \begin{aligned} R &= \frac{1}{2}(A + A^T) \\ S &= \frac{1}{2}(A - A^T) \end{aligned} \right\} \text{--- (7)}$$

$$R + S = A$$

Ex. The matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & -5 \\ 3 & -5 & 6 \end{bmatrix}$ is symmetric i.e. $a_{kj} = a_{jk}$

and $A = \begin{bmatrix} 0 & -2 & 3 \\ 2 & 0 & 4 \\ -3 & -4 & 0 \end{bmatrix}$ is skew-symmetric i.e. $a_{kj} = -a_{jk}$

(7)

while
The matrix $A = \begin{bmatrix} 2 & 3 \\ 5 & -1 \end{bmatrix}$ is neither
symmetric
nor skew-symmetric

2.5 Matrix Multiplication:

Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$

Then the product of AB is the
Matrix

$$C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

Where

$$\begin{aligned} c_{11} &= a_{11}b_{11} + a_{12}b_{21} \\ c_{12} &= a_{11}b_{12} + a_{12}b_{22} \\ c_{21} &= a_{21}b_{11} + a_{22}b_{21} \\ c_{22} &= a_{21}b_{12} + a_{22}b_{22} \end{aligned}$$

or briefly $c_{jk} = a_{j1}b_{1k} + a_{j2}b_{2k}$

$$= \sum_{i=1}^2 a_{ji}b_{ik} \dots (8)$$

Ex.1 Evaluate the following product?

$$\begin{bmatrix} 2 & 3 & 1 \\ -1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 5 & 3 \\ 4 & 0 \\ 3 & 1 \end{bmatrix} =$$

$\underbrace{\hspace{1.5cm}}_{2 \times 3} \quad \underbrace{\hspace{1.5cm}}_{3 \times 2}$

$$\begin{bmatrix} 10+12+3 & 6+0+1 \\ -5+8+12 & -3+0+4 \end{bmatrix} = \begin{bmatrix} 25 & 7 \\ 15 & 1 \end{bmatrix}$$

2×2

(8)

$$\underline{2} \quad \begin{bmatrix} 3 & 4 & 2 \\ 6 & 0 & -1 \\ -5 & -2 & 1 \end{bmatrix}_{3 \times 3} \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}_{3 \times 1} = \begin{bmatrix} 3+12+4 \\ 6+0-2 \\ -5-6+2 \end{bmatrix}$$

$$= \begin{bmatrix} 19 \\ 4 \\ -9 \end{bmatrix}_{3 \times 1}$$

$$\underline{\text{EX. 3}} \quad \begin{bmatrix} 3 & 6 & 1 \end{bmatrix}_{1 \times 3} \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}_{3 \times 1} = \begin{bmatrix} 3+12+4 \end{bmatrix}$$

$$= \begin{bmatrix} 19 \end{bmatrix}_{1 \times 1}$$

$$\underline{\text{EX. 4}} \quad \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}_{3 \times 1} \begin{bmatrix} 4 & 5 & 6 \end{bmatrix}_{1 \times 3} = \begin{bmatrix} 8 & 10 & 12 \\ 12 & 15 & 18 \\ -4 & -5 & -6 \end{bmatrix}_{3 \times 3}$$

2.5.1 Properties of Matrix Multiplication

Assuming that A, B, C are conformable for the indicated sums and products, we have

$$A(B+C) = AB+AC \quad \text{first distributive law}$$

$$(A+B)C = AC+BC \quad \text{second distributive law}$$

(9)

$A(BC) = (AB)C$ associative Law
However

$AB \neq BA$, generally

$AB = 0$ does not necessarily imply
 $A = 0$ or $B = 0$

$AB = AC$ does not necessarily imply
 $B = C$

Example:-

Let

$$A = \begin{bmatrix} 4 & 1 & 2 & 1 \\ -3 & -1 & 0 & 3 \end{bmatrix}; B = \begin{bmatrix} 3 & -1 & 4 & 1 \\ 5 & 1 & 6 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} x-1 & 3y \\ \frac{z}{2} & w-1 \\ q & p+1 \end{bmatrix}; D = \begin{bmatrix} 1 & 6 \\ 2 & 0 \\ -1 & 2 \end{bmatrix}$$

Find 1) $2A$, $-3B$, $\frac{2}{3}B$

2) $A+B$, $A-B$, $2A-3B$

3) A^T , B^T , A^T+B^T , $2A^T$, $3B^T$

If $C=D$ find the values of x, y, z, w, q & p ?

Solution:-

$$2A = \begin{bmatrix} 8 & 2 & 4 & 2 \\ -6 & -2 & 0 & 6 \end{bmatrix};$$

$$-3B = \begin{bmatrix} -9 & 3 & -12 & -3 \\ -15 & -3 & -18 & 0 \end{bmatrix}$$

$$\frac{2}{3}B = \begin{bmatrix} 2 & -\frac{2}{3} & \frac{8}{3} & \frac{2}{3} \\ \frac{10}{3} & \frac{2}{3} & 4 & 0 \end{bmatrix}$$

$$2. \quad A+B = \begin{bmatrix} 4 & 1 & 2 & 1 \\ -3 & -1 & 0 & 3 \end{bmatrix} + \begin{bmatrix} 3 & -1 & 4 & 1 \\ 5 & 1 & 6 & 0 \end{bmatrix}$$

$$\therefore A+B = \begin{bmatrix} 7 & 0 & 6 & 2 \\ 2 & 0 & 6 & 3 \end{bmatrix}$$

$$A-B = \begin{bmatrix} 1 & 2 & -2 & 0 \\ -8 & -2 & -6 & 3 \end{bmatrix}$$

$$2A-3B = \begin{bmatrix} 8 & 2 & 4 & 2 \\ -6 & -2 & 0 & 6 \end{bmatrix} - \begin{bmatrix} 9 & -3 & 12 & 3 \\ 15 & 3 & 18 & 0 \end{bmatrix}$$

$$\therefore 2A-3B = \begin{bmatrix} -1 & 5 & -8 & -1 \\ -21 & -5 & -18 & 6 \end{bmatrix}$$

$$3. \quad A^T = \begin{bmatrix} 4 & -3 \\ 1 & -1 \\ 2 & 0 \\ 1 & 3 \end{bmatrix}; B^T = \begin{bmatrix} 3 & 5 \\ -1 & 1 \\ 4 & 6 \\ 1 & 0 \end{bmatrix}$$

$$\therefore 2A^T = \begin{bmatrix} 8 & -6 \\ 2 & -2 \\ 4 & 0 \\ 2 & 6 \end{bmatrix}$$

$$\therefore A+B^T = \begin{bmatrix} 7 & 2 \\ 0 & 0 \\ -6 & 6 \\ 2 & 3 \end{bmatrix}; 3B^T = \begin{bmatrix} 9 & 15 \\ -3 & 3 \\ 12 & 18 \\ 3 & 0 \end{bmatrix}$$

$$4. \quad C=D$$

(11)

$$\begin{bmatrix} x-1 & 3y \\ \frac{z}{2} & w-1 \\ q & p+1 \end{bmatrix} = \begin{bmatrix} 1 & 6 \\ 2 & 0 \\ -1 & 2 \end{bmatrix} \Rightarrow$$

$$x-1=1 \quad \therefore x=2$$

$$3y=6 \quad \therefore y=2$$

$$\frac{z}{2}=2 \quad \therefore z=4$$

$$w-1=0 \quad \therefore w=1$$

$$q=-1 \quad \therefore q=-1$$

$$p+1=2 \quad \therefore p=1$$

Example:- Let $A = \begin{bmatrix} 1 & -3 \\ 2 & 1 \\ 4 & 0 \end{bmatrix}$; $B = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 0 & -1 \end{bmatrix}$

$$C = \begin{bmatrix} 2 & 3 \\ 7 & 1 \end{bmatrix}$$

1) Find AB 2) Find BA 3) Is $AB=BA$

4) If its possible Find CA and AC

Solution:- 1) $A \cdot B = \begin{bmatrix} 1 & -3 \\ 2 & 1 \\ 4 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 4 \\ 3 & 0 & -1 \end{bmatrix} =$

$$\begin{bmatrix} -8 & 2 & 7 \\ 5 & 4 & 7 \\ 4 & 8 & 16 \end{bmatrix}$$

2) $B \cdot A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ 2 & 1 \\ 4 & 0 \end{bmatrix}$

$$(12)$$

$$A = \begin{bmatrix} 2 & -1 \\ -1 & -9 \end{bmatrix}$$

3) $AB \neq BA$

4) $CA \rightarrow C_{2 \times 2} \cdot A_{3 \times 2}$ then

CA is not defined $2 \neq 3$

$$A \cdot C = \begin{bmatrix} 1 & -3 \\ 2 & 1 \\ 4 & 0 \end{bmatrix}_{3 \times 2} \cdot \begin{bmatrix} 2 & 3 \\ 7 & 1 \end{bmatrix}_{2 \times 2} = \begin{bmatrix} -19 & 0 \\ 11 & 7 \\ 8 & 12 \end{bmatrix}_{3 \times 2}$$

2.6 Sub Matrix :-

Any matrix obtained by omitting some rows and columns from given $(m \times n)$ matrix (A) is called a submatrix of (A) .

• Example: Submatrices of a matrix

The matrix $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$

Contains: Three (2×2) submatrices, namely

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \begin{bmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{bmatrix}, \begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix}$$

and two 1×3 submatrices (the two row vectors)

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix}, \begin{bmatrix} a_{21} & a_{22} & a_{23} \end{bmatrix}$$

It also contains;

three (2×1) Submatrices (the Column vec)

$$\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix}, \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}, \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix}$$

and six (1×2) Submatrices;

$$[a_{11} \ a_{12}], [a_{11} \ a_{13}], [a_{12} \ a_{13}]$$

$$[a_{21} \ a_{22}], [a_{21} \ a_{23}], [a_{22} \ a_{23}].$$

and six (1×1) Submatrices;

$$[a_{11}], [a_{12}], [a_{13}], [a_{21}], [a_{22}], [a_{23}]$$

" DETERMINANTS "

2-7 Definition:-

Let (A) be $(n \times n)$ matrix (square matrix), the determinants of (A) is denoted by $(\det A)$ or $|A|$ and is defined by;

$$\textcircled{1} \det(a) = |a| \quad \text{i.e. } |-2| = -2 \quad \dots (9)$$

$$\textcircled{2} \det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

$$\begin{aligned} \textcircled{3} \det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} &= \begin{vmatrix} \oplus a_{11} & \ominus a_{12} & \oplus a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\ &= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \end{aligned}$$

(14)

sample:- Evaluate $\begin{vmatrix} 2 & -4 \\ -5 & 3 \end{vmatrix}$

Solution:- $\begin{vmatrix} 2 & -4 \\ -5 & 3 \end{vmatrix} = (2)(3) - (-5)(-4) = 6 - 20 = -14$

Example:-

$$\begin{vmatrix} 2 & 1 & 3 \\ 3 & -1 & -2 \\ 2 & 3 & 1 \end{vmatrix} = 2 \begin{vmatrix} -1 & -2 \\ 3 & 1 \end{vmatrix} - 1 \begin{vmatrix} 3 & -2 \\ 2 & 1 \end{vmatrix} + 3 \begin{vmatrix} 3 & -1 \\ 2 & 3 \end{vmatrix}$$
$$= 2 [(-1)(1) - (3)(-2)] - 1 [(3)(1) - (2)(-2)] + 3 [(3)(3) - (2)(-1)]$$
$$= 2(-1 + 6) - (3 + 4) + 3(9 - (-2)) = 10 - 7 + 27 + 6$$
$$= 36$$

2-8 Minors and Cofactor

Each element a_{jk} of a determinant (A) has minors M_{jk} which is a determinant obtained by deleting the j th row and k th Column.

Example : The third order determinant

$$\det A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The minor of a_{12} is the second order determinant $M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = a_{21}a_{33} - a_{31}a_{23}$

(15)

The cofactor A_{jk} is defined as,

$$A_{jk} = (-1)^{j+k} M_{jk} \text{ ----- (10)}$$

Example : The Cofactor of a_{12} is

$$\begin{aligned} A_{12} &= (-1)^{1+2} M_{12} \\ &= (-1) \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \end{aligned}$$

OR. The sign array may be written as;

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

Now, the determinant

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

may be evaluated by $\det A = a_{11}M_{11} -$

$a_{12}M_{12} + a_{13}M_{13}$
and may be performed in five other ways
using the elements of the other two rows
and three columns.

Example : Determine the minor M_{23} of

$$\begin{bmatrix} 1 & 2 & -1 \\ 4 & 3 & -4 \\ -2 & 0 & -3 \end{bmatrix}$$

Solution :: First delete the 2nd row and 3rd Column which yields the matrix $\begin{bmatrix} 1 & 2 \\ -2 & 0 \end{bmatrix}$

Then M_{23} is the determinant of this matrix

$$M_{23} \begin{vmatrix} 1 & 2 \\ -2 & 0 \end{vmatrix} = (1)(0) - (-2)(2) = 4$$

Example : Evaluate $\begin{vmatrix} -3 & 2 & 1 \\ -5 & 0 & 6 \\ -2 & -1 & 3 \end{vmatrix}$

Solution :: Choose the row or Column where there is a number 0.

$$\begin{aligned} &= (-5)(A_{21}) + (0)(A_{22}) + (6)(A_{23}) \\ &= (-5) \begin{vmatrix} 2 & 1 \\ -1 & 3 \end{vmatrix} + (0) \begin{vmatrix} -3 & 1 \\ -2 & 3 \end{vmatrix} - (6) \begin{vmatrix} -3 & 2 \\ -2 & -1 \end{vmatrix} \\ &= 5[6+1] + 0 - 6[(-3)(-1) - (-2)(2)] \\ &= 35 + 0 - 42 = -7 \end{aligned}$$

2.9 Properties of Determinants:

(a) :: If two rows or Columns are identical, the determinant is Zero for example;

$$\begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} = 0 \quad ; \quad \begin{vmatrix} -1 & -1 \\ -3 & -3 \end{vmatrix} = 0$$

⑥: If all entries of a single row or column are zero, then the determinant is zero. for example;

$$\begin{vmatrix} 2 & -1 \\ 0 & 0 \end{vmatrix} = 0 \quad ; \quad \begin{vmatrix} 3 & 0 \\ 2 & 0 \end{vmatrix} = 0$$

⑦: Interchanging any two rows or two columns alters the det. by a factor of (-1) . for example;

$$\begin{vmatrix} 2 & 6 & 4 \\ 1 & 3 & 0 \\ -1 & 0 & 2 \end{vmatrix} = - \begin{vmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{vmatrix} = 12$$

⑧: $|A| = |A^T|$

⑨: If (A) is a diagonal matrix or triangular matrix, the $\det(A)$ is the product of the element on the main diagonal. for example;

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \quad ; \quad \begin{vmatrix} 2 & 0 \\ 0 & -1 \end{vmatrix} = -2 \quad ; \quad \begin{vmatrix} 2 & 6 & -6 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{vmatrix} =$$

$$(2)(3)(-1) = -6$$

⑩ If each element of some row or columns of a matrix (A) is multiplied by a constant (K) then the determinant is multiplied by (K) . for example;

$$\begin{vmatrix} 2 & 2 \\ 5 & -3 \end{vmatrix} = 2 \begin{vmatrix} 1 & 1 \\ 5 & -3 \end{vmatrix} \quad ; \quad \begin{vmatrix} 3 & 0 \\ -3 & 1 \end{vmatrix} = 3 \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix}$$

(18)

g) If we interchange the i th and the j th row of the a matrix (A) then the value of the determinant is change by a factor of (-1) . for example

$$54 = \begin{vmatrix} 2 & 3 & 1 \\ 4 & -2 & 0 \\ 5 & 7 & -1 \end{vmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{vmatrix} 5 & 7 & -1 \\ 4 & -2 & 0 \\ 2 & 3 & 1 \end{vmatrix} = -54$$

h) If we add (K) times the j th row to the i th row and put it in place of the i th row, then the determinant still same. for example;

$$-12 = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 0 & 1 \\ 3 & 2 & 5 \end{vmatrix} \xrightarrow{\begin{matrix} R_1 \leftrightarrow -R_2 + R_1 \\ R_3 \leftrightarrow R_2 + R_3 \end{matrix}} \begin{vmatrix} -3 & 2 & 2 \\ 4 & 0 & 1 \\ 7 & 2 & 6 \end{vmatrix} = -12$$

Example: For the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$

$$a_{11} = \begin{vmatrix} 3 & 2 \\ 3 & 4 \end{vmatrix} = 6 ; a_{12} = \begin{vmatrix} 2 & 2 \\ 3 & 4 \end{vmatrix} = -2$$

$$a_{13} = \begin{vmatrix} 2 & 3 \\ 3 & 3 \end{vmatrix} = -3 ; a_{21} = 1 ; a_{22} = -5$$

$$a_{23} = 3 ; a_{31} = -5 ; a_{32} = 4 ; a_{33} = -1$$

$$\text{and } \text{adj } A = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

$$\therefore \text{adj } A = \begin{bmatrix} 6 & 1 & -5 \\ -2 & -5 & 4 \\ -3 & 3 & -1 \end{bmatrix} \quad (19)$$

2.10 The Adjoint or Adjugate Matrix

Let $A = [a_{ij}]$ be an n -square matrix and a_{ij} be the cofactor of a_{ij} , then by definition

$$\text{adjoint } A = \text{adj } A = \begin{bmatrix} a_{11} & a_{21} & a_{31} & \dots & a_{n1} \\ a_{12} & a_{22} & \dots & \dots & a_{n2} \\ \dots & \dots & \dots & \dots & \dots \\ a_{1n} & a_{2n} & \dots & \dots & a_{nn} \end{bmatrix} \dots (11)$$

that is, the adjoint of (A) is obtained by replacing each element of (A) by its cofactor, and transposing.

Example: Let $A = \begin{bmatrix} 4 & 2 & 1 \\ 3 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$, Find its adjoint?

Solution:

$$a_{11} = \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} = -1 \quad a_{12} = (-1)^{1+2} \begin{vmatrix} 3 & 1 \\ 0 & 2 \end{vmatrix} = -6$$

$$a_{13} = (-1)^{1+3} \begin{vmatrix} 3 & 0 \\ 0 & 1 \end{vmatrix} = 3 \dots \dots \text{etc}$$

$$\text{adj } A = \begin{bmatrix} -1 & -6 & 3 \\ -3 & 8 & -4 \\ 2 & -1 & -6 \end{bmatrix}^T = \begin{bmatrix} -1 & -3 & 2 \\ -6 & 8 & -1 \\ 3 & -4 & -6 \end{bmatrix}$$

11 Inverse Matrix :-

If (A) is a square matrix of order (n) , then (A) has an inverse if and only if there is a square matrix (B) of order (n) such that $AB = I_n$

(B) is called inverse of (A) and written (A^{-1})

$$\therefore AA^{-1} = I_n \text{ and } A^{-1}A = I_n \dots (12)$$

A square matrix has an inverse if its determinant is not zero ($\neq 0$)

To find the inverse of a matrix whose determinant is not zero.

① Construct the matrix of cofactor of (A)

$$\text{Cof}(A) = C_{ij}$$

② Construct the transposed matrix of Cofactors called (the adjoint of A).

$$\text{adj}(A) = (\text{Cof } A)^T$$

$$\text{③ Then } A^{-1} = \frac{\text{adj}(A)}{\det A}$$

④ To check your answer $A \cdot A^{-1} = I$

Example :- Find the inverse of

$$A = \begin{bmatrix} 2 & 3 & -4 \\ 1 & 2 & 3 \\ 3 & -1 & -1 \end{bmatrix}$$

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$$\text{solution :- } C_{11} = (-1)^{1+1} A_{11} = \begin{vmatrix} 2 & 3 \\ -1 & -1 \end{vmatrix} = -2+3 = 1$$

$$C_{12} = (-1)^{1+2} A_{12} = - \begin{vmatrix} 1 & 3 \\ 3 & -1 \end{vmatrix} = 10$$

$$C_{13} = (-1)^{1+3} A_{13} = \begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} = -1-6 = -7$$

$$C_{21} = (-1)^{2+1} A_{21} = - \begin{vmatrix} 3 & -4 \\ -1 & -1 \end{vmatrix} = -(-3-4) = 7$$

$$C_{22} = (-1)^{2+2} A_{22} = \begin{vmatrix} 2 & -4 \\ 3 & -1 \end{vmatrix} = -2+12 = 10$$

$$C_{23} = (-1)^{2+3} A_{23} = - \begin{vmatrix} 2 & 3 \\ 3 & -1 \end{vmatrix} = -(-2-9) = 11$$

$$C_{31} = (-1)^{3+1} A_{31} = \begin{vmatrix} 3 & -4 \\ 2 & 3 \end{vmatrix} = 9+8 = 17$$

$$C_{32} = (-1)^{3+2} A_{32} = - \begin{vmatrix} 2 & -4 \\ 1 & 3 \end{vmatrix} = -(6+4) = -10$$

$$C_{33} = (-1)^{3+3} A_{33} = \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = 4-3 = 1$$

$$\text{Cof}(A) = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} = \begin{bmatrix} 1 & 10 & -7 \\ 7 & 10 & 11 \\ 17 & -10 & 1 \end{bmatrix}$$

$$\text{adj } A = (\text{Cof } A)^T = \begin{bmatrix} 1 & 7 & 17 \\ 10 & 10 & -10 \\ -7 & 11 & 1 \end{bmatrix}$$

$$\det A = |A| = \begin{vmatrix} 2 & 3 & -4 \\ 1 & 2 & 3 \\ 3 & -1 & -1 \end{vmatrix} = 2(1) + 3(10) + (-4)(-7) = 60$$

$$\therefore A^{-1} = \frac{1}{|A|} \text{adj}(A) = \frac{1}{60} \begin{bmatrix} 1 & 7 & 17 \\ 10 & 10 & -10 \\ -7 & 11 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{60} & \frac{7}{60} & \frac{17}{60} \\ \frac{10}{60} & \frac{10}{60} & \frac{-10}{60} \\ \frac{-7}{60} & \frac{11}{60} & \frac{1}{60} \end{bmatrix}$$

2.12 Linear Equations:-

Consider a system of linear equations in three unknowns x, y, z .

$$\left. \begin{aligned} a_{11}x + a_{12}y + a_{13}z &= b_1 \\ a_{21}x + a_{22}y + a_{23}z &= b_2 \\ a_{31}x + a_{32}y + a_{33}z &= b_3 \end{aligned} \right\} \dots (13)$$

In matrix relation the system of linear equations may be written as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \dots (14)$$

The above theorem called Cramer's Rule. To solve the above theorem we put;

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}; D_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}$$

$$D_2 = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}; D_3 = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}$$

If $D \neq 0$, the the system has a unique solution which is

$$x = \frac{D_1}{D}; y = \frac{D_2}{D}; z = \frac{D_3}{D} \quad (23)$$

Example:- Use Cramer's Rule to Solve

$$5x - 2y = -1$$

$$2x + 3y = 3$$

Solution:-

$$\begin{vmatrix} 5 & -2 \\ 2 & 3 \end{vmatrix} \begin{vmatrix} x \\ y \end{vmatrix} = \begin{vmatrix} -1 \\ 3 \end{vmatrix}$$

$$D = \begin{vmatrix} 5 & -2 \\ 2 & 3 \end{vmatrix} = 15 + 4 = 19$$

$$D_1 = \begin{vmatrix} -1 & -2 \\ 3 & 3 \end{vmatrix} = -3 + 6 = 3$$

$$D_2 = \begin{vmatrix} 5 & -1 \\ 2 & 3 \end{vmatrix} = 15 + 2 = 17$$

$$x = \frac{D_1}{D} = \frac{3}{19} ; y = \frac{D_2}{D} = \frac{17}{19}$$

Example:- Use Cramer's Rule to Solve

$$x + 2z = 6$$

$$-3x + 4y + 6z = 30$$

$$-x + 2y + 3z = 8$$

Solution:-

$$x = \frac{\begin{vmatrix} 6 & 0 & 2 \\ 30 & 4 & 6 \\ 8 & -2 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{vmatrix}} = \frac{-10}{11} ; y = \frac{\begin{vmatrix} 1 & 6 & 2 \\ -3 & 30 & 6 \\ -1 & 8 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{vmatrix}} = \frac{18}{11}$$

$$z = \frac{\begin{vmatrix} 1 & 0 & 6 \\ -3 & 4 & 30 \\ -1 & -2 & 8 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{vmatrix}} = \frac{38}{11}$$

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2.5 Solution of a system of linear Equations by Inverse Matrix method:-

Consider the system of (n) linear equations in (n) unknowns x_i ,

$$AX = B \quad \text{--- (15)}$$

where (A) is an $(n \times n)$ matrix $[a_{jk}]$, (X) is the column matrix (x_i) and (B) is the column matrix $[b_i]$. If (A) is non-singular. ($\det A \neq 0$), then (A^{-1}) exists and equation (15) may be written;

$$\bar{A}^{-1} A X = \bar{A}^{-1} B$$

$$I X = \bar{A}^{-1} B$$

$$\text{and } x = \bar{A}^{-1} B \quad \text{--- (16)}$$

Equation (16) is then the formal solution of equation (15).

The solution (16) may be expressed in terms of determinants and factors, As illustration Consider the case $n=3$, then

$$X = \frac{1}{\det A} (\text{adj } A) B = \frac{1}{\det A} \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$\begin{matrix} 3 \times 3 & 3 \times 1 \\ \hline \end{matrix}$

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$$\frac{1}{\det A} \begin{bmatrix} b_1 a_{11} + b_2 a_{21} + b_3 a_{31} \\ b_1 a_{12} + b_2 a_{22} + b_3 a_{32} \\ b_1 a_{13} + b_2 a_{23} + b_3 a_{33} \end{bmatrix} \dots (17)$$

Example: Solve the system by Inverse Matrix;

$$2x_1 + x_2 - x_3 = 4$$

$$x_1 - 2x_2 + x_3 = -10$$

$$-3x_1 - 2x_3 = 9$$

Solution:

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & -2 & 1 \\ -3 & 0 & -2 \end{bmatrix}; X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}; B = \begin{bmatrix} 4 \\ -10 \\ 9 \end{bmatrix}$$

Since $\det A = (-3)(-1) + (-2)(-5) = 13$

$$A^{-1} = \frac{1}{13} \begin{bmatrix} 4 & 2 & -1 \\ -1 & -7 & -3 \\ -6 & -3 & -5 \end{bmatrix}$$

$$\text{and } X = \frac{1}{13} \begin{bmatrix} 4 & 2 & -1 \\ -1 & -7 & -3 \\ -6 & -3 & -5 \end{bmatrix} \begin{bmatrix} 4 \\ -10 \\ 9 \end{bmatrix} = \frac{1}{13} \begin{bmatrix} -13 \\ 39 \\ -39 \end{bmatrix}$$

$\underbrace{\begin{bmatrix} 4 & 2 & -1 \\ -1 & -7 & -3 \\ -6 & -3 & -5 \end{bmatrix}}_{3 \times 3} \underbrace{\begin{bmatrix} 4 \\ -10 \\ 9 \end{bmatrix}}_{3 \times 1}$

Hence

$$x_1 = -1; x_2 = 3; x_3 = -3$$